Exercise 12

Consider the unusual eigenvalue problem

$$-v_{xx} = \lambda v \qquad \text{for } 0 < x < l$$

$$v_x(0) = v_x(l) = \frac{v(l) - v(0)}{l}.$$

(a) Show that $\lambda = 0$ is a double eigenvalue.

(b) Get an equation for the positive eigenvalues $\lambda > 0$.

(c) Letting $\gamma = \frac{1}{2}l\sqrt{\lambda}$, reduce the equation in part (b) to the equation

$$\gamma \sin\gamma \cos\gamma = \sin^2\gamma.$$

(d) Use part (c) to find half of the eigenvalues explicitly and half of them graphically.

(e) Assuming that all the eigenvalues are nonnegative, make a list of all the eigenfunctions.

(f) Solve the problem $u_t = ku_{xx}$ for $0 < x < l$, with the BCs given above, and with $u(x, 0) = \phi(x)$.

(g) Show that, as $t \to \infty$, $\lim u(x, t) = A + Bx$ for some constants A, B, assuming that you can take limits term by term.

Solution

Part (a)

If $\lambda = 0$, then the differential equation simplifies to

$$v_{xx} = 0.$$

Integrate both sides with respect to x.

$$v_x = C_1$$

Integrate both sides with respect to x once more to solve for $v(x)$.

$$v(x) = C_1 x + C_2$$

Now apply the boundary conditions to determine C_1 and C_2.

$$v_x(0) = v_x(l) = \frac{v(l) - v(0)}{l} \quad \rightarrow \quad C_1 = C_1 = \frac{C_1 l + C_2 - C_2}{l}$$

The boundary conditions don't tell us anything, so C_1 and C_2 remain undetermined. Since there are two arbitrary constants in the formula for $v(x)$, $\lambda = 0$ is a double eigenvalue.

Part (b)

Assuming that $\lambda > 0$, the general solution to the differential equation,

$$v_{xx} = -\lambda v,$$

can be written in terms of sine and cosine.

$$v(x) = C_3 \cos \sqrt{\lambda} x + C_4 \sin \sqrt{\lambda} x$$

Now apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} v_x(0) = v_x(l) = \frac{v(l) - v(0)}{l} \\ \rightarrow \quad C_4\sqrt{\lambda} = -C_3\sqrt{\lambda}\sin\sqrt{\lambda}l + C_4\sqrt{\lambda}\cos\sqrt{\lambda}l = \frac{C_3\cos\sqrt{\lambda}l + C_4\sin\sqrt{\lambda}l - C_3}{l} \end{aligned} \tag{1}$$

Consider the first equation in equation (1).

$$C_4\sqrt{\lambda} = -C_3\sqrt{\lambda}\sin\sqrt{\lambda}l + C_4\sqrt{\lambda}\cos\sqrt{\lambda}l$$

Solve it for C_3.

$$C_4 = -C_3 \sin\sqrt{\lambda}l + C_4\cos\sqrt{\lambda}l$$

$$C_3 \sin\sqrt{\lambda}l = (\cos\sqrt{\lambda}l - 1)C_4$$

$$C_3 = \frac{\cos\sqrt{\lambda}l - 1}{\sin\sqrt{\lambda}l}C_4$$

Now consider the second equation in equation (1).

$$C_4\sqrt{\lambda} = \frac{C_3\cos\sqrt{\lambda}l + C_4\sin\sqrt{\lambda}l - C_3}{l}$$

$$C_4\sqrt{\lambda}l = C_3\cos\sqrt{\lambda}l + C_4\sin\sqrt{\lambda}l - C_3$$

$$C_4(\sqrt{\lambda}l - \sin\sqrt{\lambda}l) = (\cos\sqrt{\lambda}l - 1)C_3$$

$$C_4 = \frac{\cos\sqrt{\lambda}l - 1}{\sqrt{\lambda}l - \sin\sqrt{\lambda}l}C_3$$

Substitute the formula for C_3 on the right side.

$$C_4 = \frac{\cos\sqrt{\lambda}l - 1}{\sqrt{\lambda}l - \sin\sqrt{\lambda}l} \cdot \frac{\cos\sqrt{\lambda}l - 1}{\sin\sqrt{\lambda}l}C_4$$

Therefore, the equation for the positive eigenvalues is

$$(\sqrt{\lambda}l - \sin\sqrt{\lambda}l)\sin\sqrt{\lambda}l = (\cos\sqrt{\lambda}l - 1)^2.$$

Part (c)

Make the substitution

$$2\gamma = \sqrt{\lambda}l \tag{2}$$

so that the equation for the positive eigenvalues becomes

$$(2\gamma - \sin 2\gamma)\sin 2\gamma = (\cos 2\gamma - 1)^2$$

$$\begin{aligned}
2\gamma \sin 2\gamma - \sin^2 2\gamma &= \cos^2 2\gamma - 2\cos 2\gamma + 1 \\
2\gamma(2\sin\gamma\cos\gamma) &= (\cos^2 2\gamma + \sin^2 2\gamma) - 2(2\cos^2\gamma - 1) + 1 \\
2\gamma(2\sin\gamma\cos\gamma) &= 2 - 2(2\cos^2\gamma - 1) \\
\gamma(2\sin\gamma\cos\gamma) &= 1 - (2\cos^2\gamma - 1) \\
2\gamma\sin\gamma\cos\gamma &= 2 - 2\cos^2\gamma \\
\gamma\sin\gamma\cos\gamma &= 1 - \cos^2\gamma.
\end{aligned}$$

Therefore,

$$\gamma\sin\gamma\cos\gamma = \sin^2\gamma.$$

Part (d)

Now solve the previous equation for γ.

$$\sin^2\gamma - \gamma\sin\gamma\cos\gamma = 0$$

$$\sin\gamma(\sin\gamma - \gamma\cos\gamma) = 0$$

$$\begin{array}{lcc}
\sin\gamma = 0 & \text{or} & \sin\gamma - \gamma\cos\gamma = 0 \\
\gamma = n\pi, \quad n = 1, 2, \ldots & \text{or} & \sin\gamma = \gamma\cos\gamma \\
 & & \tan\gamma = \gamma.
\end{array}$$

Solving equation (2) for λ

$$\lambda = \left(\frac{2\gamma}{l}\right)^2,$$

we find that half the eigenvalues are determined explicitly by

$$\boxed{\lambda = \left(\frac{2n\pi}{l}\right)^2, \quad n = 1, 2, \ldots,}$$

and the other half are determined implicitly by

$$\boxed{\lambda = \left(\frac{2\gamma_n}{l}\right)^2, \quad n = 1, 2, \ldots,}$$

where γ_n are the positive solutions to $\tan\gamma = \gamma$. Note that negative values of γ yield redundant values for λ, so only positive values of γ need to be considered.

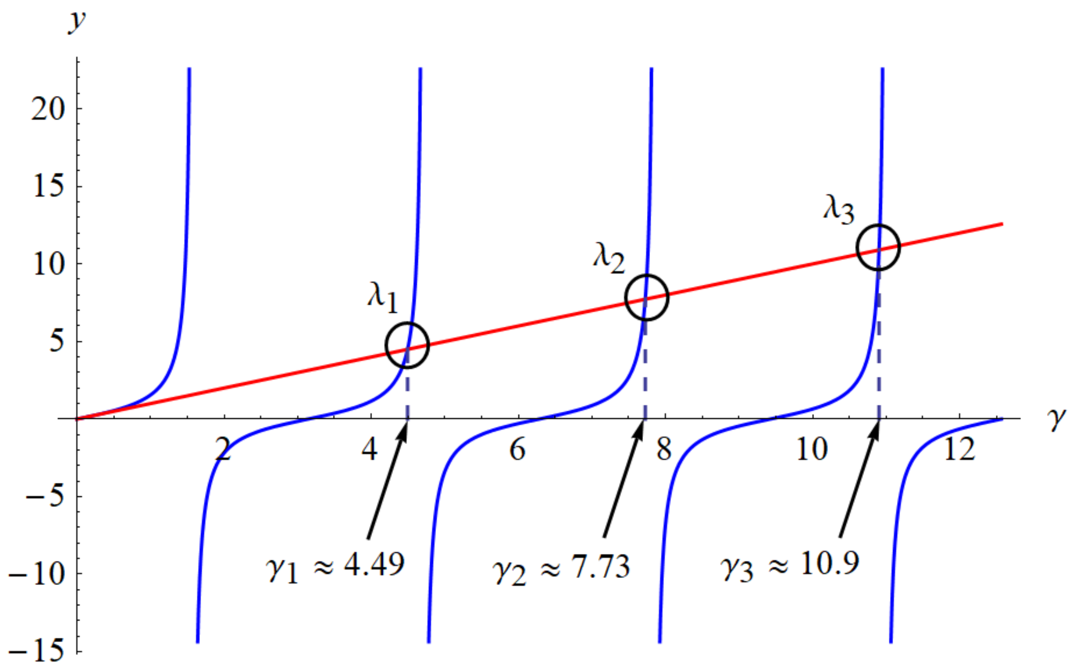

Figure 1: In blue is a plot of $y = \tan\gamma$ and in red is a plot of $y = \gamma$ for $0 \le \gamma \le 4\pi$. The eigenvalues occur wherever the graphs intersect. Because the tangent function is periodic, there are an infinite number of them.

Part (e)

If $\lambda = 0$, then $v(x) = C_1x + C_2$, so there are two eigenfunctions associated with the eigenvalue.

$$\boxed{v_1(x) = 1}$$

$$\boxed{v_2(x) = x}$$

If $\lambda = (2n\pi/l)^2$, then the eigenfunction associated with it is

$$\begin{aligned} v(x) &= C_3\cos\sqrt{\lambda}x + C_4\sin\sqrt{\lambda}x \\ &= C_3\cos\sqrt{\lambda}x + \frac{\cos\sqrt{\lambda}l - 1}{\sqrt{\lambda}l - \sin\sqrt{\lambda}l}C_3\sin\sqrt{\lambda}x \\ &= C_3\cos\frac{2n\pi x}{l} + \lim_{c\to 2n\pi}\frac{\cos c - 1}{c - \sin c}C_3\sin\frac{2n\pi x}{l} \\ &= C_3\cos\frac{2n\pi x}{l}. \end{aligned}$$

Therefore,

$$\boxed{v_3(x) = \cos\frac{2n\pi x}{l}.}$$

We had a choice initially of applying the formula for C_3 or the one for C_4. The formula for C_4 was chosen because using the one for C_3 leads to $\sin(2n\pi x/l)$, which does not satisfy the boundary conditions.

If $\lambda = (2\gamma_n/l)^2$, then the eigenfunction associated with it is

$$\begin{aligned}
v(x) &= C_3 \cos\sqrt{\lambda}x + C_4 \sin\sqrt{\lambda}x \\
&= \frac{\cos\sqrt{\lambda}l - 1}{\sin\sqrt{\lambda}l} C_4 \cos\sqrt{\lambda}x + C_4 \sin\sqrt{\lambda}x \\
&= \frac{\cos 2\gamma_n - 1}{\sin 2\gamma_n} C_4 \cos\frac{2\gamma_n x}{l} + C_4 \sin\frac{2\gamma_n x}{l} \\
&= \frac{(1 - 2\sin^2\gamma_n) - 1}{(2\sin\gamma_n\cos\gamma_n)} C_4 \cos\frac{2\gamma_n x}{l} + C_4 \sin\frac{2\gamma_n x}{l} \\
&= \frac{-2\sin^2\gamma_n}{2\sin\gamma_n\cos\gamma_n} C_4 \cos\frac{2\gamma_n x}{l} + C_4 \sin\frac{2\gamma_n x}{l} \\
&= -(\tan\gamma_n) C_4 \cos\frac{2\gamma_n x}{l} + C_4 \sin\frac{2\gamma_n x}{l} \\
&= -(\gamma_n) C_4 \cos\frac{2\gamma_n x}{l} + C_4 \sin\frac{2\gamma_n x}{l} \\
&= C_4 \left(\sin\frac{2\gamma_n x}{l} - \gamma_n \cos\frac{2\gamma_n x}{l}\right).
\end{aligned}$$

Therefore,

$$\boxed{v_4(x) = \sin\frac{2\gamma_n x}{l} - \gamma_n \cos\frac{2\gamma_n x}{l}.}$$

Part (f)

Here we will solve the PDE

$$u_t = k u_{xx},$$

subject to the boundary conditions,

$$u_x(0,t) = u_x(l,t) = \frac{u(l,t) - u(0,t)}{l},$$

and the initial condition,

$$u(x,0) = \phi(x).$$

Since the PDE and its associated boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve it. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and plug it into the PDE

$$u_t = k u_{xx} \quad \to \quad XT' = kX''T$$

and boundary conditions.

$$\begin{aligned}
u_x(0,t) = u_x(l,t) = \frac{u(l,t) - u(0,t)}{l} \\
\to \quad X'(0)T(t) = X'(l)T(t) = \frac{X(l)T(t) - X(0)T(t)}{l} \\
\to \quad X'(0) = X'(l) = \frac{X(l) - X(0)}{l}
\end{aligned}$$

Now separate variables in the PDE: bring all functions of t and constants to the left side and all functions of x to the right side. The final answer would be the same if all constants were brought to the right side.

$$\frac{T'}{kT} = \frac{X''}{X}$$

Because we have a function of t equal to a function of x, both must be equal to a constant. Let this constant be $-\lambda$ to be consistent with the previous parts of the exercise.

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

The eigenvalue problem for X was solved in parts (a) - (e). Here we will solve the differential equation for T. In the case when $\lambda = 0$ the ODE for T becomes

$$T' = 0,$$

which means $T(t)$ is just a constant. When $\lambda > 0$, the ODE for T becomes (after multiplying both sides by k)

$$\frac{T'}{T} = -k\lambda$$

Write the left side as the derivative of $\ln T$.

$$\frac{d}{dt}(\ln T) = -k\lambda$$

Integrate both sides with respect to t.

$$\ln T = -k\lambda t + C_5$$

Exponentiate both sides.

$$\begin{aligned} T(t) &= e^{-k\lambda t + C_5} \\ &= e^{-k\lambda t} e^{C_5} \end{aligned}$$

Use a new constant of integration.

$$T(t) = C_6 e^{-k\lambda t}$$

According to the principle of superposition, the solution to the PDE for u is a linear combination of the eigenfunctions $X(x)T(t)$ over all the eigenvalues. Therefore,

$$\boxed{\begin{aligned} u(x,t) = A + Bx + \sum_{n=1}^{\infty} D_n \exp\left[-k\left(\frac{2n\pi}{l}\right)^2 t\right] \cos\frac{2n\pi x}{l} \\ + \sum_{n=1}^{\infty} E_n \exp\left[-k\left(\frac{2\gamma_n}{l}\right)^2 t\right] \left(\sin\frac{2\gamma_n x}{l} - \gamma_n \cos\frac{2\gamma_n x}{l}\right), \end{aligned}}$$

where γ_n are the positive solutions to $\tan\gamma = \gamma$. The aim now is to use the provided initial condition, $u(x,0) = \phi(x)$, to determine the coefficients, A, B, D_n, and E_n.

$$u(x,0) = A + Bx + \sum_{n=1}^{\infty} D_n \cos\frac{2n\pi x}{l} + \sum_{n=1}^{\infty} E_n \left(\sin\frac{2\gamma_n x}{l} - \gamma_n \cos\frac{2\gamma_n x}{l}\right) = \phi(x) \tag{3}$$

Integrate both sides of equation (3) with respect to x from 0 to l.

$$\int_0^l \left[A + Bx + \sum_{n=1}^{\infty} D_n \cos\frac{2n\pi x}{l} + \sum_{n=1}^{\infty} E_n \left(\sin\frac{2\gamma_n x}{l} - \gamma_n \cos\frac{2\gamma_n x}{l}\right)\right] dx = \int_0^l \phi(x)\, dx$$

Split up the integral into four and bring the constants out in front.

$$A\int_0^l dx + B\int_0^l x\, dx + \sum_{n=1}^{\infty} D_n \int_0^l \cos\frac{2n\pi x}{l}\, dx + \sum_{n=1}^{\infty} E_n \int_0^l \left(\sin\frac{2\gamma_n x}{l} - \gamma_n \cos\frac{2\gamma_n x}{l}\right) dx$$
$$= \int_0^l \phi(x)\, dx$$

Evaluate the integrals.

$$A(l) + B\left(\frac{l^2}{2}\right) + \sum_{n=1}^{\infty} D_n \left[\frac{l}{2n\pi}(\overbrace{\sin 2n\pi - \sin 0}^{=0})\right] + \sum_{n=1}^{\infty} E_n \left[\frac{l \sin\gamma_n(\overbrace{-\gamma_n \cos\gamma_n + \sin\gamma_n}^{=0})}{\gamma_n}\right]$$
$$= \int_0^l \phi(x)\, dx$$

What remains is an equation relating A and B with the initial condition.

$$Al + B\frac{l^2}{2} = \int_0^l \phi(x)\, dx \tag{4}$$

To obtain another equation with A and B, multiply both sides of equation (3) by x

$$Ax + Bx^2 + \sum_{n=1}^{\infty} D_n x \cos\frac{2n\pi x}{l} + \sum_{n=1}^{\infty} E_n x \left(\sin\frac{2\gamma_n x}{l} - \gamma_n \cos\frac{2\gamma_n x}{l}\right) = x\phi(x)$$

and then integrate both sides with respect to x from 0 to l.

$$\int_0^l \left[Ax + Bx^2 + \sum_{n=1}^{\infty} D_n x \cos\frac{2n\pi x}{l} + \sum_{n=1}^{\infty} E_n x \left(\sin\frac{2\gamma_n x}{l} - \gamma_n \cos\frac{2\gamma_n x}{l}\right)\right] dx = \int_0^l x\phi(x)\, dx$$

Split up the integral on the left into four and bring the constants in front of them.

$$A\int_0^l x\, dx + B\int_0^l x^2\, dx + \sum_{n=1}^{\infty} D_n \int_0^l x\cos\frac{2n\pi x}{l}\, dx + \sum_{n=1}^{\infty} E_n \int_0^l x\left(\sin\frac{2\gamma_n x}{l} - \gamma_n \cos\frac{2\gamma_n x}{l}\right) dx$$
$$= \int_0^l x\phi(x)\, dx$$

Evaluate the integrals.

$$A\left(\frac{l^2}{2}\right) + B\left(\frac{l^3}{3}\right) + \sum_{n=1}^{\infty} D_n \left[\frac{l^2(-1 + \overbrace{\cos 2n\pi}^{=1} + \overbrace{2n\pi \sin 2n\pi}^{=0})}{4n^2\pi^2}\right]$$
$$+ \sum_{n=1}^{\infty} E_n \left\{\frac{-l^2[\overbrace{-\gamma_n + 3\gamma_n \cos 2\gamma_n + (2\gamma_n^2 - 1)\sin 2\gamma_n}^{=0}]}{4\gamma_n^2}\right\} = \int_0^l x\phi(x)\, dx$$

Proof that $-\gamma_n + 3\gamma_n \cos 2\gamma_n + (2\gamma_n^2 - 1)\sin 2\gamma_n = 0$ will be given at the end. What remains is a second equation relating A and B with the initial condition.

$$A\frac{l^2}{2} + B\frac{l^3}{3} = \int_0^l x\phi(x)\,dx \tag{5}$$

Solving the system of equations, (4) and (5), yields

$$A = \frac{1}{l^2}\left[4l\int_0^l \phi(x)\,dx - 6\int_0^l x\phi(x)\,dx\right] \quad \text{and} \quad B = \frac{1}{l^3}\left[12\int_0^l x\phi(x)\,dx - 6l\int_0^l \phi(x)\,dx\right].$$

Therefore,

$$\boxed{A = \frac{2}{l^2}\int_0^l (2l - 3x)\phi(x)\,dx} \quad \text{and} \quad \boxed{B = \frac{6}{l^3}\int_0^l (2x - l)\phi(x)\,dx.}$$

To obtain an equation for D_n, multiply both sides of equation (3) by $\cos(2m\pi x/l)$, where m is an integer,

$$A\cos\frac{2m\pi x}{l} + Bx\cos\frac{2m\pi x}{l} + \sum_{n=1}^\infty D_n \cos\frac{2m\pi x}{l}\cos\frac{2n\pi x}{l} + \sum_{n=1}^\infty E_n \cos\frac{2m\pi x}{l}\left(\sin\frac{2\gamma_n x}{l} - \gamma_n\cos\frac{2\gamma_n x}{l}\right) = \phi(x)\cos\frac{2m\pi x}{l}$$

and then integrate both sides with respect to x from 0 to l.

$$\int_0^l \left[A\cos\frac{2m\pi x}{l} + Bx\cos\frac{2m\pi x}{l} + \sum_{n=1}^\infty D_n \cos\frac{2m\pi x}{l}\cos\frac{2n\pi x}{l} + \sum_{n=1}^\infty E_n \cos\frac{2m\pi x}{l}\left(\sin\frac{2\gamma_n x}{l} - \gamma_n\cos\frac{2\gamma_n x}{l}\right)\right]dx = \int_0^l \phi(x)\cos\frac{2m\pi x}{l}\,dx$$

Split up the integral into four and bring the constants out in front.

$$A\int_0^l \cos\frac{2m\pi x}{l}\,dx + B\int_0^l x\cos\frac{2m\pi x}{l}\,dx + \sum_{n=1}^\infty D_n\int_0^l \cos\frac{2m\pi x}{l}\cos\frac{2n\pi x}{l}\,dx + \sum_{n=1}^\infty E_n\int_0^l \cos\frac{2m\pi x}{l}\left(\sin\frac{2\gamma_n x}{l} - \gamma_n\cos\frac{2\gamma_n x}{l}\right)dx = \int_0^l \phi(x)\cos\frac{2m\pi x}{l}\,dx$$

Evaluate the first two integrals.

$$A\left[\frac{l}{2m\pi}(\overbrace{\sin 2m\pi}^{=0})\right] + B\left[\frac{l^2(-1 + \overbrace{\cos 2m\pi}^{=1} + \overbrace{2m\pi\sin 2m\pi)}^{=0}}{4m^2\pi^2}\right] + \sum_{n=1}^\infty D_n\int_0^l \cos\frac{2m\pi x}{l}\cos\frac{2n\pi x}{l}\,dx + \sum_{n=1}^\infty E_n\int_0^l \cos\frac{2m\pi x}{l}\left(\sin\frac{2\gamma_n x}{l} - \gamma_n\cos\frac{2\gamma_n x}{l}\right)dx = \int_0^l \phi(x)\cos\frac{2m\pi x}{l}\,dx$$

The remaining integrals on the left side are equal to zero (as can be verified with the product-to-sum formulas for sine and cosine) except for the term in the first series when $n = m$.

$$D_n \int_0^l \cos^2 \frac{2n\pi x}{l}\, dx = \int_0^l \phi(x) \cos \frac{2n\pi x}{l}\, dx$$

Evaluate the last integral on the left.

$$D_n \left(\frac{l}{2}\right) = \int_0^l \phi(x) \cos \frac{2n\pi x}{l}\, dx$$

Therefore,

$$\boxed{D_n = \frac{2}{l} \int_0^l \phi(x) \cos \frac{2n\pi x}{l}\, dx.}$$

To obtain a similar formula for E_n, multiply both sides of equation (3) by $F_m(x) = \sin(2\gamma_m x/l) - \gamma_m \cos(2\gamma_m x/l)$, where m is an integer.

$$AF_m(x) + BxF_m(x) + \sum_{n=1}^{\infty} D_n F_m(x) \cos \frac{2n\pi x}{l} + \sum_{n=1}^{\infty} E_n F_m(x) F_n(x) = \phi(x) F_m(x)$$

and then integrate both sides with respect to x from 0 to l.

$$\int_0^l \left[AF_m(x) + BxF_m(x) + \sum_{n=1}^{\infty} D_n F_m(x) \cos \frac{2n\pi x}{l} + \sum_{n=1}^{\infty} E_n F_m(x) F_n(x)\right] dx = \int_0^l \phi(x) F_m(x)\, dx$$

Split up the integral into four and bring the constants out in front.

$$A \int_0^l F_m(x)\, dx + B \int_0^l xF_m(x)\, dx + \sum_{n=1}^{\infty} D_n \int_0^l F_m(x) \cos \frac{2n\pi x}{l}\, dx \\ + \sum_{n=1}^{\infty} E_n \int_0^l F_m(x) F_n(x)\, dx = \int_0^l \phi(x) F_m(x)\, dx$$

Evaluate the first two integrals.

$$A\left[\frac{l \sin \gamma_m \overbrace{(-\gamma_m \cos \gamma_m + \sin \gamma_m)}^{=0}}{\gamma_m}\right] + B\left\{-\frac{l^2 \overbrace{[-\gamma_m + 3\gamma_m \cos 2\gamma_m + (2\gamma_m^2 - 1) \sin 2\gamma_m]}^{=0}}{4\gamma_m^2}\right\} \\ + \sum_{n=1}^{\infty} D_n \int_0^l F_m(x) \cos \frac{2n\pi x}{l}\, dx + \sum_{n=1}^{\infty} E_n \int_0^l F_m(x) F_n(x)\, dx = \int_0^l \phi(x) F_m(x)\, dx$$

The remaining integrals on the left side are equal to zero (as can be verified with the product-to-sum formulas for sine and cosine) except for the term in the second series when $n = m$.

$$E_n \int_0^l F_n^2(x)\, dx = \int_0^l \phi(x) F_n(x)\, dx$$

Evaluate the last integral on the left.

$$E_n \left[\frac{2l\gamma_n(1 + 2\gamma_n^2 + \cos 4\gamma_n) + l(\gamma_n^2 - 1) \sin 4\gamma_n}{8\gamma_n}\right] = \int_0^l \phi(x) F_n(x)\, dx$$

Therefore,

$$\boxed{E_n = \frac{8\gamma_n}{2l\gamma_n(1+2\gamma_n^2+\cos 4\gamma_n)+l(\gamma_n^2-1)\sin 4\gamma_n}\int_0^l \phi(x)\left(\sin\frac{2\gamma_n x}{l}-\gamma_n\cos\frac{2\gamma_n x}{l}\right)dx,}$$

where γ_n are the positive solutions to $\tan\gamma = \gamma$. Finally, it will be shown that $\gamma - 3\gamma\cos 2\gamma + (1-2\gamma^2)\sin 2\gamma = 0$.

$$\begin{aligned}
\gamma - 3\gamma\cos 2\gamma + (1-2\gamma^2)\sin 2\gamma &= \gamma - 3\gamma(1-2\sin^2\gamma) + (1-2\gamma^2)(2\sin\gamma\cos\gamma) \\
&= \gamma - 3\gamma + 6\gamma\sin^2\gamma + 2\sin\gamma\cos\gamma - 4\gamma^2\sin\gamma\cos\gamma \\
&= -2\gamma + 6\tan\gamma\sin^2\gamma + 2\sin\gamma\cos\gamma - 4\tan^2\gamma\sin\gamma\cos\gamma \\
&= -2\tan\gamma + 6\frac{\sin^3\gamma}{\cos\gamma} + 2\sin\gamma\cos\gamma - 4\frac{\sin^3\gamma}{\cos\gamma} \\
&= -2\frac{\sin\gamma}{\cos\gamma} + 2\sin\gamma\cos\gamma + 2\frac{\sin^3\gamma}{\cos\gamma} \\
&= -2\frac{\sin\gamma}{\cos\gamma} + 2\frac{\sin\gamma}{\cos\gamma}\cos^2\gamma + 2\frac{\sin^3\gamma}{\cos\gamma} \\
&= 2\frac{\sin\gamma}{\cos\gamma}(-1+\cos^2\gamma) + 2\frac{\sin^3\gamma}{\cos\gamma} \\
&= 2\frac{\sin\gamma}{\cos\gamma}(-\sin^2\gamma) + 2\frac{\sin^3\gamma}{\cos\gamma} \\
&= -2\frac{\sin^3\gamma}{\cos\gamma} + 2\frac{\sin^3\gamma}{\cos\gamma} \\
&= 0
\end{aligned}$$

Part (g)

Because of the exponential functions, each of the series in the solution for $u(x,t)$ tends to 0 as $t \to \infty$.

$$\begin{aligned}\lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}\left\{A + Bx + \sum_{n=1}^{\infty} D_n \exp\left[-k\left(\frac{2n\pi}{l}\right)^2 t\right]\cos\frac{2n\pi x}{l}\right.\\ &\qquad \left. + \sum_{n=1}^{\infty} E_n \exp\left[-k\left(\frac{2\gamma_n}{l}\right)^2 t\right]\left(\sin\frac{2\gamma_n x}{l} - \gamma_n \cos\frac{2\gamma_n x}{l}\right)\right\}\\ &= A + Bx + \sum_{n=1}^{\infty} D_n \left\{\lim_{t\to\infty}\exp\left[-k\left(\frac{2n\pi}{l}\right)^2 t\right]\right\}\cos\frac{2n\pi x}{l}\\ &\qquad + \sum_{n=1}^{\infty} E_n \left\{\lim_{t\to\infty}\exp\left[-k\left(\frac{2\gamma_n}{l}\right)^2 t\right]\right\}\left(\sin\frac{2\gamma_n x}{l} - \gamma_n \cos\frac{2\gamma_n x}{l}\right)\\ &= A + Bx\end{aligned}$$

Substitute the formulas found for A and B (using s for the dummy integration variable).

$$\begin{aligned}\lim_{t\to\infty} u(x,t) &= \frac{2}{l^2}\int_0^l (2l-3s)\phi(s)\,ds + \left[\frac{6}{l^3}\int_0^l (2s-l)\phi(s)\,ds\right]x\\ &= \frac{2}{l^2}\int_0^l (2l-3s)\phi(s)\,ds + \frac{6x}{l^3}\int_0^l (2s-l)\phi(s)\,ds\\ &= \frac{2}{l^3}\left[l\int_0^l (2l-3s)\phi(s)\,ds + 3x\int_0^l (2s-l)\phi(s)\,ds\right]\end{aligned}$$

Therefore, the equilibrium (steady-state) solution is

$$\boxed{\lim_{t\to\infty} u(x,t) = \frac{2}{l^3}\int_0^l [l(2l-3s) + 3x(2s-l)]\phi(s)\,ds.}$$