Exercise 13

Consider a string that is fixed at the end $x = 0$ and is free at the end $x = l$ except that a load (weight) of given mass is attached to the right end.

(a) Show that it satisfies the problem

$$u_{tt} = c^2 u_{xx} \qquad \text{for } 0 < x < l$$
$$u(0,t) = 0 \qquad u_{tt}(l,t) = -k u_x(l,t)$$

for some constant k.

(b) What is the eigenvalue problem in this case?

(c) Find the equation for the positive eigenvalues and find the eigenfunctions.

Solution

Part (a)

The integral formulation will be used to obtain the equation of motion for the string.

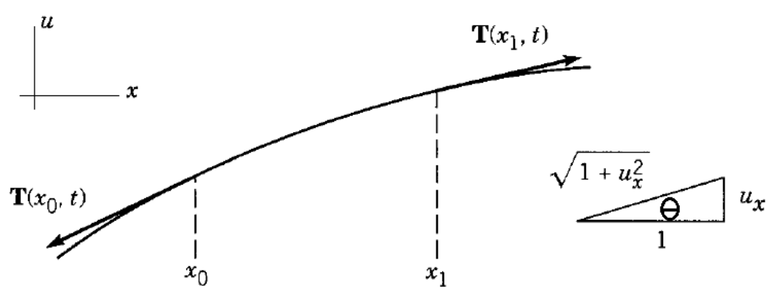

Figure 1: Schematic of the string.

In order to derive the equation of motion, we will invoke Newton's second law, which states that the sum of the forces acting on a body is equal to its mass times its acceleration. Mathematically this is written as

$$\sum \mathbf{F} = m\mathbf{a}.$$

Note that this is a vector equation; in other words, there is a separate equation for each component of force and corresponding component of acceleration. For this problem we will choose the coordinate system as shown in the figure, so there are two equations of significance.

$$\sum F_x = ma_x$$
$$\sum F_u = ma_u$$

There are two forces acting on this string, $\mathbf{T}$ at $x = x_0$ and $\mathbf{T}$ at $x = x_1$. The motion of the string is entirely vertical, which means there is no horizontal component of acceleration ($a_x = 0$). Hence, the resistive force of the medium will only act vertically. The tensions, on the other hand, have components in both the x-direction and u-direction and have to be resolved using cosine and sine, respectively.

Horizontal component of $\mathbf{T}$ at $x = x_0$:	$-T\cos\theta_0$
Horizontal component of $\mathbf{T}$ at $x = x_1$:	$+T\cos\theta_1$
Vertical component of $\mathbf{T}$ at $x = x_0$:	$-T\sin\theta_0$
Vertical component of $\mathbf{T}$ at $x = x_1$:	$+T\sin\theta_1$,

where θ_0 and θ_1 are the angles between the vectors and the x-axis at x_0 and x_1, respectively. To determine θ it is necessary to note that $\tan\theta$ is equal to rise over run, the slope. If the height of the string is $u(x,t)$, the slope is given by $\partial u/\partial x = u_x$. As shown in Figure 1, the hypotenuse can be determined using Pythagorean's theorem. And now $\cos\theta$ can be written in terms of u.
Newton's second law in the x-direction is thus

$$\sum F_x = -T\cos\theta_0 + T\cos\theta_1 = ma_x = 0$$

$$-T\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x=x_0} + T\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x=x_1} = 0.$$

Hence,

$$T\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x=x_0} = T\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x=x_1}.$$

This equation can be simplified if we make the assumption that u, and therefore u_x, remains small for all x and t. The binomial theorem tells us that

$$\sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}u_x^4 + \cdots.$$

Compared to 1, u_x^2 and all higher powers of u_x can be considered negligible. Approximating the square root terms as 1, the equation of motion simplifies to

$$T|_{x=x_0} \approx T|_{x=x_1}.$$

This equation tells us that the magnitude of the tension at x_0 is equal to the magnitude of the tension at x_1; that is, T is constant. Let's move on to Newton's second law in the u-direction. If the height of the string is $u(x,t)$, the rate of change of the height with respect to time, the velocity, is given by $\partial u/\partial t = u_t$. Consequently, the rate of change of velocity with respect to time, the acceleration, is $\partial^2 u/\partial t^2 = u_{tt}$. Note that u_{tt} is the acceleration at a specific point on the string, x, at time t. To get the force we therefore have to multiply u_{tt} by a tiny bit of mass dm. Mass, of course, is density times length, so this can be written in terms of arc length, s, as $dm = \rho\, ds$. The total force is obtained by integrating $u_{tt}\, dm$ over the mass of the string. Newton's second law in the u-direction is thus

$$\sum F_u = -T\sin\theta_0 + T\sin\theta_1 = \int_{\text{mass of string}} u_{tt}\, dm.$$

Write θ in terms of u using the right triangle in Figure 1 and substitute $dm = \rho\, ds$.

$$-T\left.\frac{u_x}{\sqrt{1+u_x^2}}\right|_{x=x_0} + T\left.\frac{u_x}{\sqrt{1+u_x^2}}\right|_{x=x_1} = \int_{\text{length of string}} \rho u_{tt}\, ds = \int_{x_0}^{x_1} \rho u_{tt}\sqrt{1+u_x^2}\, dx$$

This equation can be simplified if we make the assumption that u, and therefore u_x, remains small for all x and t. The binomial theorem tells us that

$$\sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}u_x^4 + \cdots .$$

Compared to 1, u_x^2 and all higher powers of u_x can be considered negligible. Approximating the square root terms as 1, the equation of motion simplifies to

$$Tu_x(x_1,t) - Tu_x(x_0,t) \approx \int_{x_0}^{x_1} \rho u_{tt}\,dx.$$

According to the fundamental theorem of calculus,

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

so the left side can be written as

$$Tu_x(x_1,t) - Tu_x(x_0,t) = \int_{x_0}^{x_1} \frac{\partial}{\partial x}(Tu_x)\,dx.$$

Hence,

$$\int_{x_0}^{x_1} \frac{\partial}{\partial x}(Tu_x)\,dx = \int_{x_0}^{x_1} \rho u_{tt}\,dx$$

$$\int_{x_0}^{x_1} \left[\frac{\partial}{\partial x}(Tu_x)\right]\,dx = \int_{x_0}^{x_1} \rho u_{tt}\,dx.$$

Thus, the integrands must be equal to each other.

$$\frac{\partial}{\partial x}(Tu_x) = \rho u_{tt}$$

The tension is constant, so

$$Tu_{xx} = \rho u_{tt}.$$

Therefore,

$$u_{tt} = c^2 u_{xx},$$

where $c^2 = T/\rho$. The left end at $x = 0$ is fixed, which means u is constant there. We choose this constant to equal 0:

$$u(0,t) = 0.$$

The boundary condition at the right end ($x = l$) has to be determined by making a free body diagram and subsequently using Newton's second law.

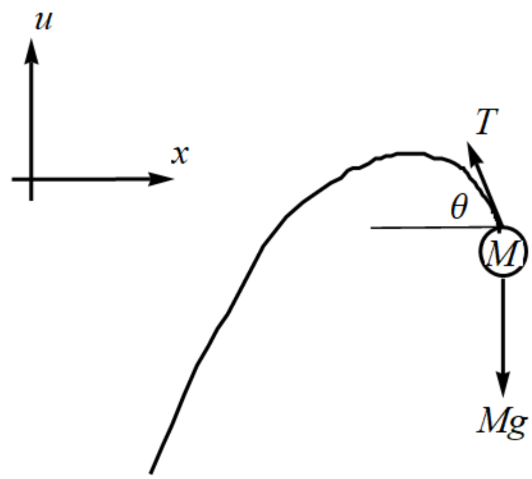

Figure 2: Free body diagram of the mass M attached at the right end of the string.

Applying Newton's second law in the u-direction at the mass M, we get

$$T\sin\theta|_{x=l} - Mg = Mu_{tt}(l,t).$$

Write θ in terms of u using Figure 1.

$$T\left.\frac{u_x}{\sqrt{1+u_x^2}}\right|_{x=l} - Mg = Mu_{tt}(l,t).$$

With the assumption that oscillations are small, $\sqrt{1+u_x^2} \approx 1$, and the equation of motion simplifies to

$$Tu_x(l,t) - Mg \approx Mu_{tt}(l,t).$$

Factor $Tu_x(l,t)$ from the left side.

$$Tu_x(l,t)\left[1 - \underbrace{\frac{Mg}{Tu_x(l,t)}}_{\approx 0}\right] = Mu_{tt}(l,t)$$

Here we make the assumption that the mass M is really small compared to the tension so that the fraction can be neglected. Doing so results in a boundary condition that is homogeneous.

$$Tu_x(l,t) \approx Mu_{tt}(l,t)$$

At $x = l$, therefore,

$$u_{tt}(l,t) = -ku_x(l,t),$$

where $-k = T/M$.

Part (b)

Since the PDE and its boundary conditions are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and plug it into the PDE

$$u_{tt} = c^2u_{xx} \quad \rightarrow \quad XT'' = c^2X''T$$

and the boundary conditions.

$$\begin{array}{rclcl} u(0,t)=0 & \to & X(0)T(t)=0 & \to & X(0)=0 \\ u_{tt}(l,t)=-ku_x(l,t) & \to & X(l)T''(t)=-kX'(l)T(t) & & \end{array}$$

Now separate variables in the PDE: bring all functions of t and constants to the left side and all functions of x to the right side. The final answer would be the same if all constants were brought to the right side.

$$\frac{T''}{c^2T}=\frac{X''}{X}$$

Because we have a function of t equal to a function of x for all t and x, both must be equal to a constant. Let this constant be $-\lambda$.

$$\frac{T''}{c^2T}=\frac{X''}{X}=-\lambda$$

As a result, the second boundary condition becomes

$$X(l)T''(t)=-kX'(l)T(t) \quad\to\quad \frac{T''(t)}{T(t)}=-k\frac{X'(l)}{X(l)} \quad\to\quad -\lambda c^2=-k\frac{X'(l)}{X(l)}$$

Therefore, the eigenvalue problem to solve is

$$-X''=\lambda X$$

subject to the boundary conditions,

$$X(0)=0 \quad\text{and}\quad kX'(l)=\lambda c^2X(l).$$

Part (c)

Assuming λ is positive ($\lambda=\beta^2$), the ODE for X becomes

$$X''=-\beta^2X.$$

The general solution can be written in terms of sine and cosine.

$$X(x)=C_1\cos\beta x+C_2\sin\beta x$$

Apply the boundary conditions here to determine C_1 and C_2.

$$\begin{array}{rcr} X(0)=0 & \to & C_1=0 \\ kX'(l)=\lambda c^2X(l) & \to & k\beta C_2\cos\beta l=\lambda c^2C_2\sin\beta l \end{array}$$

The second equation is a transcendental equation for β.

$$k\beta C_2\cos\beta l=\beta^2c^2C_2\sin\beta l$$

Divide both sides by $\beta^2c^2C_2$. Therefore, the eigenvalues are $\lambda=\beta^2$, where β satisfies

$$\frac{k}{\beta c^2}=\tan\beta l.$$

The eigenfunctions associated with them are

$$\begin{aligned} X(x)&=C_1\cos\beta x+C_2\sin\beta x \\ &=C_2\sin\beta x. \end{aligned}$$

Therefore,

$$X(x)=\sin\beta x.$$