Exercise 14

Solve the eigenvalue problem $x^2u'' + 3xu' + \lambda u = 0$ for $1 < x < e$, with $u(1) = u(e) = 0$. Assume that $\lambda > 1$. (*Hint:* Look for solutions of the form $u = x^m$ for complex m.)

Solution

The ODE is equidimensional, so the solution is of the form,

$$u = x^m.$$

In order to determine m, we will plug this form into the ODE. Before doing this, though, find out what u' and u'' are.

$$u = x^m \quad \rightarrow \quad u' = mx^{m-1} \quad \rightarrow \quad u'' = m(m-1)x^{m-2}$$

Now we're ready to make the substitution.

$$x^2 m(m-1)x^{m-2} + 3xmx^{m-1} + \lambda x^m = 0$$

$$m(m-1)x^m + 3mx^m + \lambda x^m = 0$$

Divide both sides by x^m.

$$m(m-1) + 3m + \lambda = 0$$

What remains is a quadratic equation for m.

$$m^2 + 2m + \lambda = 0$$

Use the quadratic formula to obtain its solution.

$$\begin{aligned} m &= \frac{-2 \pm \sqrt{4 - 4\lambda}}{2} \\ &= -1 \pm \sqrt{1 - \lambda} \end{aligned}$$

Since λ is assumed to be greater than 1, m is complex. Factor -1 and bring i out of the square root

$$m = -1 \pm i\sqrt{\lambda - 1}$$

so that now the square root yields a real number. The solution to the ODE is thus

$$u(x) = C_1 x^{-1} \cos\left(\sqrt{\lambda - 1}\ln x\right) + C_2 x^{-1} \sin\left(\sqrt{\lambda - 1}\ln x\right).$$

Now apply the two boundary conditions to determine C_1 and C_2.

$$\begin{aligned} u(1) &= C_1 = 0 \\ u(e) &= C_1 e^{-1} \cos\left(\sqrt{\lambda - 1}\right) + C_2 e^{-1} \sin\left(\sqrt{\lambda - 1}\right) = 0 \end{aligned}$$

The second equation simplifies to $C_2 e^{-1} \sin(\sqrt{\lambda - 1}) = 0$. To obtain a nontrivial solution for u, we insist that $C_2 \neq 0$. An equation for λ results from this.

$$\sin(\sqrt{\lambda - 1}) = 0.$$

$$\sqrt{\lambda - 1} = n\pi, \quad n = 1, 2, \ldots$$

Therefore,

$$\lambda = \lambda_n = n^2\pi^2 + 1.$$

The eigenfunctions associated with these eigenvalues are

$$u(x) = C_2 x^{-1} \sin\left(\sqrt{\lambda - 1} \ln x\right) \quad \to \quad u_n(x) = x^{-1} \sin(n\pi \ln x).$$