Exercise 16

Find the positive eigenvalues and the corresponding eigenfunctions of the fourth-order operator $+d^4/dx^4$ with the four boundary conditions

$$X(0) = X(l) = X''(0) = X''(l) = 0.$$

Solution

The eigenvalue problem we have to solve here is

$$\frac{d^4}{dx^4}X = \lambda X$$

with the four boundary conditions in the problem statement. Since we only care about the positive eigenvalues, let $\lambda = \mu^4$. We use μ^4 and not μ^2 in order to make the solution to the ODE more convenient to work with.

$$\frac{d^4 X}{dx^4} = \mu^4 X$$

This is a linear homogeneous ODE with constant coefficients, so the solution is of the form,

$$X = e^{rx}.$$

This form will be substituted into the equation to determine r. Before doing so, write the derivatives of X first.

$$X = e^{rx} \quad \rightarrow \quad X' = re^{rx} \quad \rightarrow \quad X'' = r^2 e^{rx} \quad \rightarrow \quad X''' = r^3 e^{rx} \quad \rightarrow \quad X'''' = r^4 e^{rx}$$

Now we're ready to make the substitutions.

$$r^4 e^{rx} = \mu^4 e^{rx}$$

Divide both sides by e^{rx}.

$$r^4 = \mu^4$$

Solve this equation for r by factoring.

$$r^4 - \mu^4 = 0 \quad \rightarrow \quad (r^2 - \mu^2)(r^2 + \mu^2) = 0$$

So we have

$$r = \{\pm\mu, \pm i\mu\}.$$

The solution to the ODE is thus

$$X(x) = C_1 e^{-\mu x} + C_2 e^{\mu x} + C_3 \cos \mu x + C_4 \sin \mu x.$$

Apply the four boundary conditions now to determine C_1, C_2, C_3, and C_4.

$$X(0) = C_1 + C_2 + C_3 = 0 \tag{1}$$
$$X''(0) = \mu^2(C_1 + C_2 - C_3) = 0 \tag{2}$$
$$X(l) = C_1 e^{-\mu l} + C_2 e^{\mu l} + C_3 \cos \mu l + C_4 \sin \mu l = 0 \tag{3}$$
$$X''(l) = \mu^2(C_1 e^{-\mu l} + C_2 e^{\mu l} - C_3 \cos \mu l - C_4 \sin \mu l) = 0 \tag{4}$$

Solve equation (2) for C_3

$$C_3 = C_1 + C_2$$

and plug it into equation (1).

$$C_1 + C_2 + (C_1 + C_2) = 0 \quad \rightarrow \quad 2(C_1 + C_2) = 0 \quad \rightarrow \quad C_2 = -C_1$$

This implies that $C_3 = 0$. Consequently, equations (3) and (4) become

$$\begin{aligned} C_1 e^{-\mu l} - C_1 e^{\mu l} + C_4 \sin \mu l = 0 \\ C_1 e^{-\mu l} - C_1 e^{\mu l} - C_4 \sin \mu l = 0 \end{aligned}$$

Adding these two equations yields

$$2C_1(e^{-\mu l} - e^{\mu l}) = 0 \quad \rightarrow \quad C_1 = 0,$$

and subtracting these two equations yields

$$2C_4 \sin \mu l = 0.$$

In order to avoid the trivial solution, we insist that $C_4 \neq 0$. Then we have the following equation for μ.

$$\sin \mu l = 0$$

Solve it for μl.

$$\mu l = n\pi, \quad n = 1, 2, \ldots$$

So then

$$\mu = \mu_n = \frac{n\pi}{l}, \quad n = 1, 2, \ldots.$$

Therefore, the eigenvalues are

$$\lambda = \mu^4 = \frac{n^4 \pi^4}{l^4}, \quad n = 1, 2, \ldots,$$

and the eigenfunctions associated with them are

$$X(x) = C_4 \sin \mu x \quad \rightarrow \quad X_n(x) = \sin \frac{n\pi x}{l}.$$