Exercise 17

Solve the fourth-order eigenvalue problem $X'''' = \lambda X$ in $0 < x < l$, with the four boundary conditions

$$X(0) = X'(0) = X(l) = X'(l) = 0,$$

where $\lambda > 0$. (*Hint:* First solve the fourth-order ODE.)

Solution

Since we only care about the positive eigenvalues, let $\lambda = \mu^4$. We use μ^4 and not μ^2 in order to make the solution to the ODE more convenient to work with.

$$X'''' = \mu^4 X$$

This is a linear homogeneous ODE with constant coefficients, so the solution is of the form,

$$X = e^{rx}.$$

This form will be substituted into the equation to determine r. Before doing so, write the derivatives of X first.

$$X = e^{rx} \quad \rightarrow \quad X' = re^{rx} \quad \rightarrow \quad X'' = r^2 e^{rx} \quad \rightarrow \quad X''' = r^3 e^{rx} \quad \rightarrow \quad X'''' = r^4 e^{rx}$$

Now we're ready to make the substitutions.

$$r^4 e^{rx} = \mu^4 e^{rx}$$

Divide both sides by e^{rx}.

$$r^4 = \mu^4$$

Solve this equation for r by factoring.

$$r^4 - \mu^4 = 0 \quad \rightarrow \quad (r^2 - \mu^2)(r^2 + \mu^2) = 0$$

So we have

$$r = \{\pm\mu, \pm i\mu\}.$$

The solution to the ODE is thus

$$X(x) = C_1 e^{-\mu x} + C_2 e^{\mu x} + C_3 e^{-i\mu x} + C_4 e^{i\mu x}.$$

With the help of Euler's formula,

$$e^{i\mu x} = \cos \mu x + i \sin \mu x,$$

it can be written in terms of trigonometric functions. This is done again for the sake of convenience.

$$X(x) = C_1(\cos i\mu x + i \sin i\mu x) + C_2(\cos i\mu x - i \sin i\mu x) + C_3(\cos \mu x - i \sin \mu x) + C_4(\cos \mu x + i \sin \mu x)$$

Use the facts that $\cosh \mu x = \cos i\mu x$ and $\sinh \mu x = -i \sin i\mu x$.

$$= C_1(\cosh \mu x - \sinh \mu x) + C_2(\cosh \mu x + \sinh \mu x) + C_3(\cos \mu x - i \sin \mu x) + C_4(\cos \mu x + i \sin \mu x)$$
$$= (C_1 + C_2) \cosh \mu x + (-C_1 + C_2) \sinh \mu x + (C_3 + C_4) \cos \mu x + (-iC_3 + iC_4) \sin \mu x$$

Introduce new constants of integration here.

$$X(x) = C_5 \cosh \mu x + C_6 \sinh \mu x + C_7 \cos \mu x + C_8 \sin \mu x$$

Apply the four boundary conditions now to determine C_5, C_6, C_7, and C_8.

$$X(0) = C_5 + C_7 = 0 \tag{1}$$
$$X'(0) = \mu(C_6 + C_8) = 0 \tag{2}$$
$$X(l) = C_5 \cosh \mu l + C_6 \sinh \mu l + C_7 \cos \mu l + C_8 \sin \mu l = 0 \tag{3}$$
$$X'(l) = \mu(C_5 \sinh \mu l + C_6 \cosh \mu l - C_7 \sin \mu l + C_8 \cos \mu l) = 0 \tag{4}$$

From equations (1) and (2) we find that $C_5 = -C_7$ and $C_6 = -C_8$. Plug these into equations (3) and (4).

$$-C_7 \cosh \mu l - C_8 \sinh \mu l + C_7 \cos \mu l + C_8 \sin \mu l = 0 \tag{5}$$
$$-C_7 \sinh \mu l - C_8 \cosh \mu l - C_7 \sin \mu l + C_8 \cos \mu l = 0 \tag{6}$$

In order to avoid the trivial solution, we insist that $C_7 \neq 0$ and $C_8 \neq 0$. Solve equation (5) for C_7

$$C_7 = \frac{\sinh \mu l - \sin \mu l}{-\cosh \mu l + \cos \mu l} C_8$$

and substitute it into equation (6).

$$-\frac{\sinh \mu l - \sin \mu l}{-\cosh \mu l + \cos \mu l} C_8 \sinh \mu l - C_8 \cosh \mu l - \frac{\sinh \mu l - \sin \mu l}{-\cosh \mu l + \cos \mu l} C_8 \sin \mu l + C_8 \cos \mu l = 0$$

Divide both sides by $-C_8$.

$$\frac{\sinh \mu l - \sin \mu l}{-\cosh \mu l + \cos \mu l} \sinh \mu l + \cosh \mu l + \frac{\sinh \mu l - \sin \mu l}{-\cosh \mu l + \cos \mu l} \sin \mu l - \cos \mu l = 0$$

$$-\frac{\sinh \mu l - \sin \mu l}{\cosh \mu l - \cos \mu l} \sinh \mu l + \cosh \mu l - \frac{\sinh \mu l - \sin \mu l}{\cosh \mu l - \cos \mu l} \sin \mu l - \cos \mu l = 0$$

$$-\frac{\sinh \mu l - \sin \mu l}{\cosh \mu l - \cos \mu l} (\sinh \mu l + \sin \mu l) + \cosh \mu l - \cos \mu l = 0$$

Multiply both sides by $\cosh \mu l - \cos \mu l$.

$$-(\sinh \mu l - \sin \mu l)(\sinh \mu l + \sin \mu l) + (\cosh \mu l - \cos \mu l)^2 = 0$$

Expand the left side.

$$-\sinh^2 \mu l + \sin^2 \mu l + \cosh^2 \mu l - 2 \cosh \mu l \cos \mu l + \cos^2 \mu l = 0$$

Use the facts that $\cosh^2 \mu l - \sinh^2 \mu l = 1$ and $\sin^2 \mu l + \cos^2 \mu l = 1$.

$$2 - 2 \cosh \mu l \cos \mu l = 0$$

Therefore, the eigenvalues are

$$\lambda = \mu^4,$$

where μ is obtained from the transcendental equation,

$$\cosh \mu l \cos \mu l = 1.$$

The eigenfunctions associated with these eigenvalues are

$$\begin{aligned}
X(x) &= C_5 \cosh \mu x + C_6 \sinh \mu x + C_7 \cos \mu x + C_8 \sin \mu x \\
&= -C_7 \cosh \mu x - C_8 \sinh \mu x + C_7 \cos \mu x + C_8 \sin \mu x \\
&= C_7(-\cosh \mu x + \cos \mu x) + C_8(-\sinh \mu x + \sin \mu x) \\
&= \frac{\sinh \mu l - \sin \mu l}{-\cosh \mu l + \cos \mu l} C_8(-\cosh \mu x + \cos \mu x) + C_8(-\sinh \mu x + \sin \mu x) \\
&= \frac{-C_8}{-\cosh \mu l + \cos \mu l}[(\sinh \mu l - \sin \mu l)(\cosh \mu x - \cos \mu x) + (-\cosh \mu l + \cos \mu l)(\sinh \mu x - \sin \mu x)] \\
&= C_9[(\sinh \mu l - \sin \mu l)(\cosh \mu x - \cos \mu x) - (\cosh \mu l - \cos \mu l)(\sinh \mu x - \sin \mu x)].
\end{aligned}$$

Therefore,

$$X(x) = (\sinh \mu l - \sin \mu l)(\cosh \mu x - \cos \mu x) - (\cosh \mu l - \cos \mu l)(\sinh \mu x - \sin \mu x).$$