Exercise 18

A tuning fork may be regarded as a pair of vibrating flexible bars with a certain degree of stiffness. Each such bar is clamped at one end and is approximately modeled by the fourth-order PDE $u_{tt} + c^2 u_{xxxx} = 0$. It has initial conditions as for the wave equation. Let's say that on the end $x = 0$ it is clamped (fixed), meaning that it satisfies $u(0,t) = u_x(0,t) = 0$. On the other end $x = l$ it is free, meaning that it satisfies $u_{xx}(l,t) = u_{xxx}(l,t) = 0$. Thus there are a total of four boundary conditions, two at each end.

(a) Separate the time and space variables to get the eigenvalue problem $X'''' = \lambda X$.

(b) Show that zero is not an eigenvalue.

(c) Assuming that all the eigenvalues are positive, write them as $\lambda = \beta^4$ and find the equation for β.

(d) Find the frequencies of vibration.

(e) Compare your answer in part (d) with the overtones of the vibrating string by looking at the ratio β_2^2/β_1^2. Explain why you hear an almost pure tone when you listen to a tuning fork.

Solution

The PDE,

$$u_{tt} + c^2 u_{xxxx} = 0,$$

and the boundary conditions,

$$u(0,t) = u_x(0,t) = u_{xx}(l,t) = u_{xxx}(l,t) = 0,$$

are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form, $u(x,t) = T(t)X(x)$, and plug it into the PDE,

$$XT'' + c^2 X''''T = 0,$$

and the boundary conditions,

$$\begin{aligned} u(0,t) = X(0)T(t) = 0 \quad &\rightarrow \quad X(0) = 0 \\ u_x(0,t) = X'(0)T(t) = 0 \quad &\rightarrow \quad X'(0) = 0 \\ u_{xx}(l,t) = X''(l)T(t) = 0 \quad &\rightarrow \quad X''(l) = 0 \\ u_{xxx}(l,t) = X'''(l)T(t) = 0 \quad &\rightarrow \quad X'''(l) = 0. \end{aligned}$$

Separate variables now.

$$-\frac{T''}{c^2 T} = \frac{X''''}{X}$$

Note that the minus sign and c^2 can go on either side; it will not affect the final answer for $u(x,t)$. We have a function of t on the left and a function of x on the right. Since they are equal to each other for all x and t, they must be equal to a constant.

$$-\frac{T''}{c^2 T} = \frac{X''''}{X} = \lambda$$

Values of λ for which $X(0) = 0$ and $X'(0) = 0$ and $X''(l) = 0$ and $X'''(l) = 0$ are satisfied are called the eigenvalues. The nontrivial functions $X(x)$ associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \beta^4$

We choose λ to be β^4 rather than β^2 because of the fourth derivative—it makes the solution more convenient to work with. The differential equation for X becomes

$$\frac{X''''}{X} = \beta^4.$$

Multiply both sides by X.

$$X'''' = \beta^4 X$$

This is a linear homogeneous ODE with constant coefficients, so the solution is of the form,

$$X = e^{rx}.$$

This form will be substituted into the equation to determine r. Before doing so, write the derivatives of X first.

$$X = e^{rx} \quad \rightarrow \quad X' = re^{rx} \quad \rightarrow \quad X'' = r^2 e^{rx} \quad \rightarrow \quad X''' = r^3 e^{rx} \quad \rightarrow \quad X'''' = r^4 e^{rx}$$

Now we're ready to make the substitutions.

$$r^4 e^{rx} = \beta^4 e^{rx}$$

Divide both sides by e^{rx}.

$$r^4 = \beta^4$$

Solve this equation for r by factoring.

$$r^4 - \beta^4 = 0 \quad \rightarrow \quad (r^2 - \beta^2)(r^2 + \beta^2) = 0$$

So we have

$$r = \{\pm\beta, \pm i\beta\}.$$

The solution to the ODE is thus

$$X(x) = C_1 e^{-\beta x} + C_2 e^{\beta x} + C_3 e^{-i\beta x} + C_4 e^{i\beta x}.$$

With the help of Euler's formula,

$$e^{i\beta x} = \cos\beta x + i\sin\beta x,$$

it can be written in terms of trigonometric functions. This is done again for the sake of convenience.

$$X(x) = C_1(\cos i\beta x + i\sin i\beta x) + C_2(\cos i\beta x - i\sin i\beta x) + C_3(\cos\beta x - i\sin\beta x) + C_4(\cos\beta x + i\sin\beta x)$$

Use the facts that $\cosh\beta x = \cos i\beta x$ and $\sinh\beta x = -i\sin i\beta x$.

$$\begin{aligned} &= C_1(\cosh\beta x - \sinh\beta x) + C_2(\cosh\beta x + \sinh\beta x) + C_3(\cos\beta x - i\sin\beta x) + C_4(\cos\beta x + i\sin\beta x) \\ &= (C_1 + C_2)\cosh\beta x + (-C_1 + C_2)\sinh\beta x + (C_3 + C_4)\cos\beta x + (-iC_3 + iC_4)\sin\beta x \end{aligned}$$

Introduce new constants of integration here.

$$X(x) = C_5 \cosh \beta x + C_6 \sinh \beta x + C_7 \cos \beta x + C_8 \sin \beta x$$

Apply the four boundary conditions now to determine C_5, C_6, C_7, and C_8.

$$X(0) = C_5 + C_7 = 0 \tag{1}$$
$$X'(0) = \beta(C_6 + C_8) = 0 \tag{2}$$
$$X''(l) = \beta^2(C_5 \cosh \beta l + C_6 \sinh \beta l - C_7 \cos \beta l - C_8 \sin \beta l) = 0 \tag{3}$$
$$X'''(l) = \beta^3(C_5 \sinh \beta l + C_6 \cosh \beta l + C_7 \sin \beta l - C_8 \cos \beta l) = 0 \tag{4}$$

Solve equations (1) and (2) for C_5 and C_6.

$$C_5 = -C_7$$
$$C_6 = -C_8$$

Substitute these results into equations (3) and (4).

$$-C_7 \cosh \beta l - C_8 \sinh \beta l - C_7 \cos \beta l - C_8 \sin \beta l = 0 \tag{5}$$
$$-C_7 \sinh \beta l - C_8 \cosh \beta l + C_7 \sin \beta l - C_8 \cos \beta l = 0 \tag{6}$$

Solve equation (5) for C_7

$$C_7 = -\frac{\sinh \beta l + \sin \beta l}{\cosh \beta l + \cos \beta l} C_8$$

and plug it into equation (6).

$$\frac{\sinh \beta l + \sin \beta l}{\cosh \beta l + \cos \beta l} C_8 \sinh \beta l - C_8 \cosh \beta l - \frac{\sinh \beta l + \sin \beta l}{\cosh \beta l + \cos \beta l} C_8 \sin \beta l - C_8 \cos \beta l = 0$$

Divide both sides by C_8.

$$\frac{\sinh \beta l + \sin \beta l}{\cosh \beta l + \cos \beta l} \sinh \beta l - \cosh \beta l - \frac{\sinh \beta l + \sin \beta l}{\cosh \beta l + \cos \beta l} \sin \beta l - \cos \beta l = 0$$

Factor the left side.

$$\frac{\sinh \beta l + \sin \beta l}{\cosh \beta l + \cos \beta l} (\sinh \beta l - \sin \beta l) - (\cosh \beta l + \cos \beta l) = 0$$

Multiply both sides by $\cosh \beta l + \cos \beta l$.

$$(\sinh \beta l + \sin \beta l)(\sinh \beta l - \sin \beta l) - (\cosh \beta l + \cos \beta l)^2 = 0$$

Expand the left side.

$$\sinh^2 \beta l - \sin^2 \beta l - \cosh^2 \beta l - 2 \cosh \beta l \cos \beta l - \cos^2 \beta l = 0$$

Use the facts that $\cosh^2 \beta l - \sinh^2 \beta l = 1$ and $\sin^2 \beta l + \cos^2 \beta l = 1$.

$$-2 - 2 \cosh \beta l \cos \beta l = 0$$

Therefore, the positive eigenvalues are $\lambda = \beta^4$, where β satisfies

$$\cosh \beta l \cos \beta l + 1 = 0.$$

The eigenfunctions associated with these eigenvalues are

$$\begin{aligned}X(x) &= C_5 \cosh \beta x + C_6 \sinh \beta x + C_7 \cos \beta x + C_8 \sin \beta x \\ &= -C_7 \cosh \beta x - C_8 \sinh \beta x + C_7 \cos \beta x + C_8 \sin \beta x \\ &= -C_7(\cosh \beta x - \cos \beta x) - C_8(\sinh \beta x - \sin \beta x) \\ &= \frac{\sinh \beta l + \sin \beta l}{\cosh \beta l + \cos \beta l} C_8(\cosh \beta x - \cos \beta x) - C_8(\sinh \beta x - \sin \beta x) \\ &= \frac{C_8}{\cosh \beta l + \cos \beta l}[(\sinh \beta l + \sin \beta l)(\cosh \beta x - \cos \beta x) - (\cosh \beta l + \cos \beta l)(\sinh \beta x - \sin \beta x)] \\ &= C_0[(\sinh \beta l + \sin \beta l)(\cosh \beta x - \cos \beta x) - (\cosh \beta l + \cos \beta l)(\sinh \beta x - \sin \beta x)].\end{aligned}$$

Therefore,

$$X(x) = (\sinh \beta l + \sin \beta l)(\cosh \beta x - \cos \beta x) - (\cosh \beta l + \cos \beta l)(\sinh \beta x - \sin \beta x).$$

Now the ODE for T will be solved.

$$-\frac{T''}{c^2 T} = \beta^4$$

Multiply both sides by $-c^2 T$.

$$T'' = -c^2 \beta^4 T$$

The general solution can be written in terms of sine and cosine.

$$T(t) = D_1 \cos c\beta^2 t + D_2 \sin c\beta^2 t$$

Determination of the Zero Eigenvalue: $\lambda = 0$

Assuming $\lambda = 0$, the differential equation for X becomes

$$\frac{X''''}{X} = 0.$$

Multiply both sides by X.

$$X'''' = 0$$

Integrate both sides with respect to x.

$$X'''(x) = C_9$$

Apply the fourth boundary condition here to determine C_9.

$$X'''(l) = C_9 = 0$$

Integrate both sides of the ODE with respect to x again.

$$X''(x) = C_{10}$$

Apply the third boundary condition here to determine C_{10}.

$$X''(l) = C_{10} = 0$$

Integrate both sides of the ODE with respect to x again.

$$X'(x) = C_{11}$$

Apply the second boundary condition here to determine C_{11}.

$$X'(0) = C_{11} = 0$$

Integrate both sides of the ODE with respect to x for the last time.

$$X(x) = C_{12}$$

Apply the first boundary condition here to determine C_{12}.

$$X(0) = C_{12} = 0$$

Therefore, we obtain the trivial solution for $X(x)$,

$$X(x) = 0,$$

which means zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\eta^4$

The negative eigenvalues can be disregarded on the physical grounds that they lead to solutions that blow up or decay exponentially in time. That is, the ODE for $T(t)$ becomes

$$-\frac{T''}{c^2T} = -\eta^4.$$

Multiply both sides by $-c^2T$.

$$T'' = c^2\eta^4 T$$

The solution can be written in terms of the exponential function.

$$T(t) = Ae^{-c\eta^2 t} + Be^{c\eta^2 t}$$

Solving the ODE for X is quite long and unnecessary; only a brief solution will be presented here to confirm the point.

$$\frac{X''''}{X} = -\eta^4$$

Its solution is

$$X(x) = \exp\left(\frac{\eta}{\sqrt{2}}x\right)\left[C_{17}\cos\left(\frac{\eta}{\sqrt{2}}x\right) + C_{18}\sin\left(\frac{\eta}{\sqrt{2}}x\right)\right] + \exp\left(-\frac{\eta}{\sqrt{2}}x\right)\left[C_{19}\cos\left(\frac{\eta}{\sqrt{2}}x\right) + C_{20}\sin\left(\frac{\eta}{\sqrt{2}}x\right)\right].$$

Applying the four boundary conditions yields the following equation for η:

$$\frac{(3C_{19} - C_{20})e^{\sqrt{2}\eta l} + C_{19} + C_{20}}{(C_{19} - C_{20})e^{\sqrt{2}\eta l} + C_{19} - C_{20}} = \tan\left(\frac{\eta}{\sqrt{2}}l\right), \tag{7}$$

where

$$C_{19} = \frac{C_{20}}{2 - \frac{2}{1+e^{\sqrt{2}\eta l}} + \tan\left(\frac{\eta}{\sqrt{2}}l\right)}.$$

The intersections of the functions on both sides of equation (7) are the solutions for η.

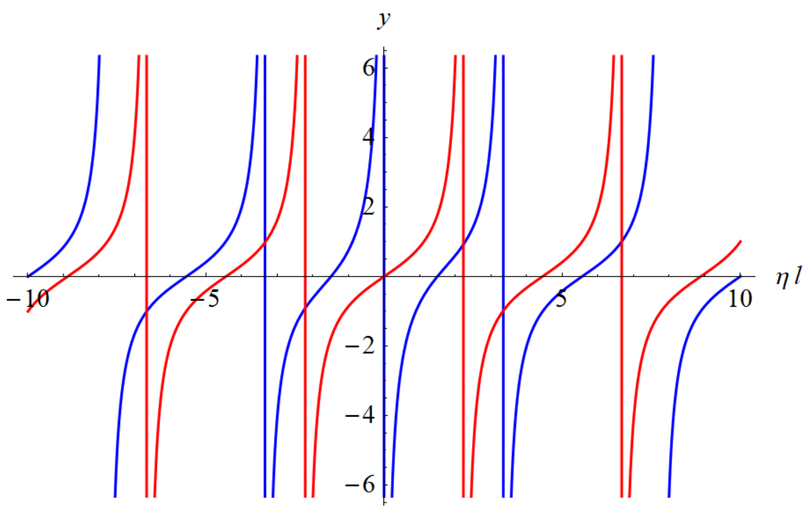

Figure 1: In blue is a plot of the function on the left side of equation (7) as a function of ηl and in red is a plot of the function on the right side of equation (7) as a function of ηl. Since the graphs do not intersect, there are no negative eigenvalues. The vertical lines are the asymptotes of the functions; they show that the curves do not intersect outside the viewing window.

Part (d)

According to the principle of linear superposition, the solution to the PDE for $u(x,t)$ is a linear combination of all products $T_n(t)X_n(x)$ over all the eigenvalues.

$$u(x,t) = \sum_{n=1}^{\infty}(A_n \cos c\beta_n^2 t + B_n \sin c\beta_n^2 t)\times$$
$$[(\sinh\beta_n l + \sin\beta_n l)(\cosh\beta_n x - \cos\beta_n x) - (\cosh\beta_n l + \cos\beta_n l)(\sinh\beta_n x - \sin\beta_n x)],$$

where β_n is the nth zero of the function $\cosh\beta l \cos\beta l + 1$ (see Figure 2). If we had two initial conditions, the coefficients, A_n and B_n, could be determined. The frequencies of vibration in radians per second are given by the coefficients of t in the arguments of sine and cosine:

$$\omega_n = c\beta_n^2.$$

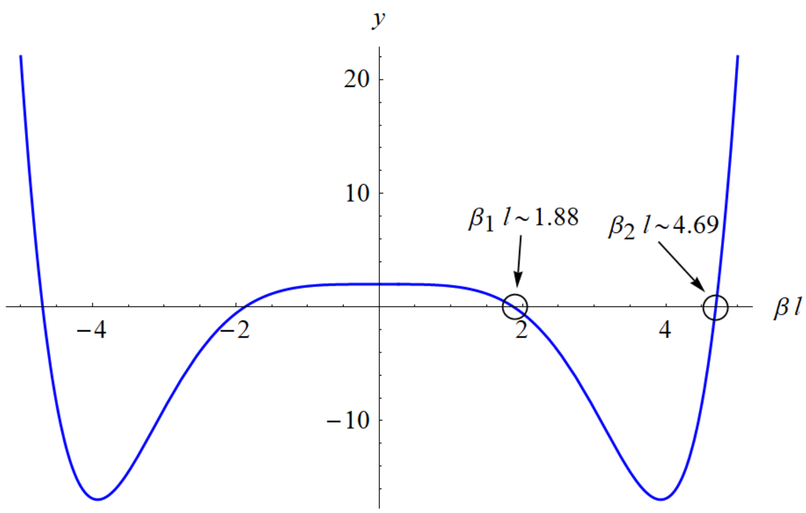

Figure 2: This is a plot of the function $\cosh \beta l \cos \beta l + 1$ as a function of βl. Since the eigenvalues are given by $\lambda = \beta^4$ and hyperbolic cosine and cosine are even functions, negative values of βl give redundant values for λ.

Part (e)

As discussed in section 4.1, the frequencies of a vibrating string are given by

$$\frac{n\pi}{l}\sqrt{\frac{T}{\rho}}, \quad n = 1, 2, \ldots.$$

All the overtones are multiples of the fundamental frequency at $n = 1$. However, for a tuning fork

$$\begin{aligned}\frac{\omega_2}{\omega_1} &= \frac{c\beta_2^2}{c\beta_1^2}\\ &= \frac{\beta_2^2}{\beta_1^2}\\ &\approx \frac{(4.69/l)^2}{(1.88/l)^2}\\ &\approx 6.27\end{aligned}$$

we find that the first overtone has a frequency that is about 6.27 times higher than the fundamental. When a tuning fork is struck, most of the energy goes into the fundamental mode of vibration,

$$X_1(x) = (\sinh \beta_1 l + \sin \beta_1 l)(\cosh \beta_1 x - \cos \beta_1 x) - (\cosh \beta_1 l + \cos \beta_1 l)(\sinh \beta_1 x - \sin \beta_1 x),$$

with little of it going to the higher modes $[X_2(x), X_3(x), \ldots]$. As a result, the tuning fork gives an almost pure note. For a plucked string the energy is distributed more evenly among the different modes of vibration. Because of this, the sound from a string is a result of many frequencies and is not pure.