Exercise 4

Consider the Robin eigenvalue problem. If

$$a_0 < 0, \quad a_l < 0 \quad \text{and} \quad -a_0 - a_l < a_0 a_l l,$$

show that there are *two* negative eigenvalues. This case may be called "substantial absorption at both ends." (*Hint:* Show that the rational curve $y = -(a_0 + a_l)\gamma/(\gamma^2 + a_0 a_l)$ has a single maximum and crosses the line $y = 1$ in two places. Deduce that it crosses the tanh curve in two places.)

Solution

The Robin eigenvalue problem is the ODE,

$$-X'' = \lambda X$$

subject to the boundary conditions

$$\begin{aligned} X'(0) - a_0 X(0) &= 0 \\ X'(l) + a_l X(l) &= 0 \end{aligned}$$

The negative eigenvalues are given by $\lambda = -\gamma^2$, where γ was shown in the previous exercise to satisfy

$$\tanh \gamma l = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}.$$

The hyperbolic tangent function never goes higher than $y = 1$, so if we can show that the rational function has a maximum above the $y = 1$ line and crosses it in two places, then it will cross the tanh curve in two places as well. Setting the rational function equal to 1, we obtain a quadratic function for γ.

$$-\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l} = 1$$

$$-(a_0 + a_l)\gamma = \gamma^2 + a_0 a_l$$

$$\gamma^2 + (a_0 + a_l)\gamma + a_0 a_l = 0$$

Use the quadratic formula to solve for γ.

$$\begin{aligned} \gamma &= \frac{-(a_0 + a_l) \pm \sqrt{(a_0 + a_l)^2 - 4a_0 a_l}}{2} \\ &= \frac{-(a_0 + a_l) \pm \sqrt{(a_0 - a_l)^2}}{2} \\ &= \frac{-a_0 - a_l \pm (a_0 - a_l)}{2} \end{aligned}$$

Hence, the rational function intersects the $y = 1$ line at two values, $\gamma = -a_l$ and $\gamma = -a_0$. Next we will show that the rational function has a maximum above the line. This is done by taking the derivative of it with respect to γ and setting it equal to 0. Use the quotient rule.

$$\frac{d}{d\gamma}\left[-\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}\right] = \frac{-(a_0 + a_l)(\gamma^2 + a_0 a_l) + 2\gamma^2(a_0 + a_l)}{(\gamma^2 + a_0 a_l)^2} = 0$$

Solving this equation for γ, we can find where the extrema occur. The numerator must be equal to 0.

$$-(a_0 + a_l)(\gamma^2 + a_0 a_l) + 2\gamma^2(a_0 + a_l) = 0$$

Divide both sides by $a_0 + a_l$.

$$-(\gamma^2 + a_0 a_l) + 2\gamma^2 = 0$$

Solve for γ by factoring.

$$\gamma^2 - a_0 a_l = 0$$

$$(\gamma + \sqrt{a_0 a_l})(\gamma - \sqrt{a_0 a_l}) = 0$$

So the extrema occur where $\gamma = \pm\sqrt{a_0 a_l}$. To find which of these the maximum is at, we evaluate the second derivative of the rational function at each of these values of γ.

$$\frac{d^2}{d\gamma^2}\left[-\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}\right]\bigg|_{\gamma=\sqrt{a_0 a_l}} = \frac{\sqrt{a_0 a_l}(a_0 + a_l)}{2(a_0 a_l)^2}$$

$$\frac{d^2}{d\gamma^2}\left[-\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}\right]\bigg|_{\gamma=-\sqrt{a_0 a_l}} = -\frac{\sqrt{a_0 a_l}(a_0 + a_l)}{2(a_0 a_l)^2}$$

Since a_0 and a_l are both negative, the second derivative at $\gamma = \sqrt{a_0 a_l}$ is negative and the second derivative at $\gamma = -\sqrt{a_0 a_l}$ is positive. By the Second Derivative Test, this means a maximum is located at $\gamma = \sqrt{a_0 a_l}$ and a minimum is located at $\gamma = -\sqrt{a_0 a_l}$. Therefore, the rational curve $y = -(a_0 + a_l)\gamma/(\gamma^2 + a_0 a_l)$ has a single maximum and crosses the line $y = 1$ in two places. Because of this, the rational function crosses the tanh curve in two places, meaning there are two negative eigenvalues.

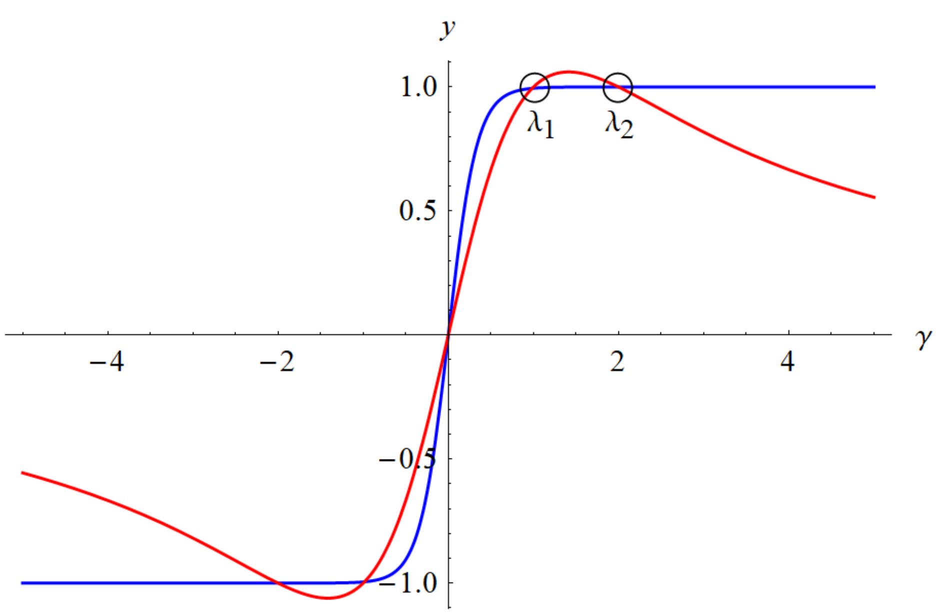

Figure 1: This is a plot of the rational function (in red) and the hyperbolic tangent function (in blue) for $a_0 = -1$, $a_l = -2$, and $l = 3$. Since the eigenvalues are given by $\lambda = -\gamma^2$ and both functions are odd, negative values of γ give redundant values for λ, so only intersections at positive values of γ need to be considered.