Exercise 5

In Exercise 4 (substantial absorption at both ends) show graphically that there are an infinite number of positive eigenvalues. Show graphically that they satisfy (11) and (12).

Solution

The positive eigenvalues of the Robin problem are given by $\lambda = \beta^2$, where β satisfies equation (10) in the text.

$$\tan \beta l = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l} \tag{10}$$

Substantial absorption at both ends occurs when

$$a_0 < 0, \quad a_l < 0, \quad \text{and } -a_0 - a_l < a_0 a_l l.$$

The plot of these two functions looks like Figure 1 in the textbook on page 95, but since a_0 and a_l are negative here, the rational function is reflected about the β-axis. Also, because $-a_0 - a_l < a_0 a_l l$ (as opposed to $-a_0 - a_l > a_0 a_l l$), the rational function does not intersect the first branch of the tangent curve $[0 < \beta < \pi/(2l)]$.

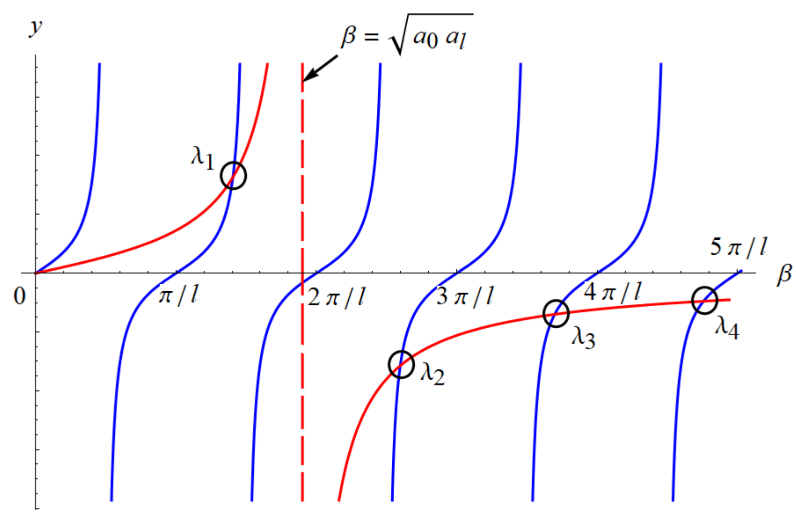

Figure 1: This is a plot of $y = \tan \beta l$ (in blue) and $y = [(a_0 + a_l)\beta]/(\beta^2 - a_0 a_l)$ in red for $a_0 < 0$ and $a_l < 0$ and $-a_0 - a_l < a_0 a_l l$. The line $\beta = \sqrt{a_0 a_l}$ is the vertical asymptote of the rational function. The zeroes of the tangent function occur at $n\pi/l$, where n is an integer.

From Figure 1 we can see there are infinitely many positive eigenvalues because both functions extend to the right indefinitely. Equation (11) is satisfied but with n starting from 1 rather than 0.

$$\frac{n^2\pi^2}{l^2} < \lambda_n < \frac{(n+1)^2\pi^2}{l^2}, \quad n = 1, 2, \ldots$$

Equation (12) is satisfied as well with a slight modification, $n + 1$ rather than n, because the rational function intersects the tangent function below the β-axis for $\beta > \sqrt{a_0 a_l}$.

$$\lim_{n \to \infty} \left[\beta_n - (n+1)\frac{\pi}{l}\right] = 0, \quad n = 1, 2, \ldots$$