Exercise 9

On the interval $0 \leq x \leq 1$ of length one, consider the eigenvalue problem

$$-X'' = \lambda X$$
$$X'(0) + X(0) = 0 \qquad \text{and} \qquad X(1) = 0$$

(absorption at one end and zero at the other).

(a) Find an eigenfunction with eigenvalue zero. Call it $X_0(x)$.

(b) Find an equation for the positive eigenvalues $\lambda = \beta^2$.

(c) Show graphically from part (b) that there are an infinite number of positive eigenvalues.

(d) Is there a negative eigenvalue?

Solution

Part (a)

If $\lambda = 0$, then the ODE simplifies to

$$X'' = 0.$$

Integrate both sides with respect to x.

$$X' = C_1$$

Integrate both sides with respect to x once more to solve for $X(x)$.

$$X(x) = C_1 x + C_2$$

Apply the provided boundary conditions now to determine C_1 and C_2.

$$X'(0) + X(0) = C_1 + C_2 = 0$$
$$X(1) = C_1 + C_2 = 0$$

From these equations we have $C_2 = -C_1$, so the eigenfunction is

$$X(x) = C_1 x - C_1$$
$$= C_1(x - 1)$$

Therefore,

$$X_0(x) = x - 1.$$

Part (b)

Assuming the eigenvalues are positive ($\lambda = \beta^2$), the differential equation becomes

$$X'' = -\beta^2 X.$$

Its general solution can be written in terms of sines and cosines.

$$X(x) = C_3 \cos \beta x + C_4 \sin \beta x$$

Apply the provided boundary conditions now to determine C_3 and C_4.

$$\begin{aligned} X'(0) + X(0) &= C_3 + C_4\beta = 0 \\ X(1) &= C_3 \cos\beta + C_4 \sin\beta = 0 \end{aligned}$$

From the first equation we have $C_3 = -C_4\beta$. Plug this into the second equation to obtain

$$-C_4\beta\cos\beta + C_4\sin\beta = 0$$

$$C_4\sin\beta = C_4\beta\cos\beta.$$

Therefore, the positive eigenvalues are given by $\lambda = \beta^2$, where β satisfies the transcendental equation,

$$\tan\beta = \beta.$$

Part (c)

Negative values of β yield redundant values for λ, so only positive values of β need to be considered. The solutions to $\tan\beta = \beta$ are where the curves, $y = \beta$ and $y = \tan\beta$, intersect.

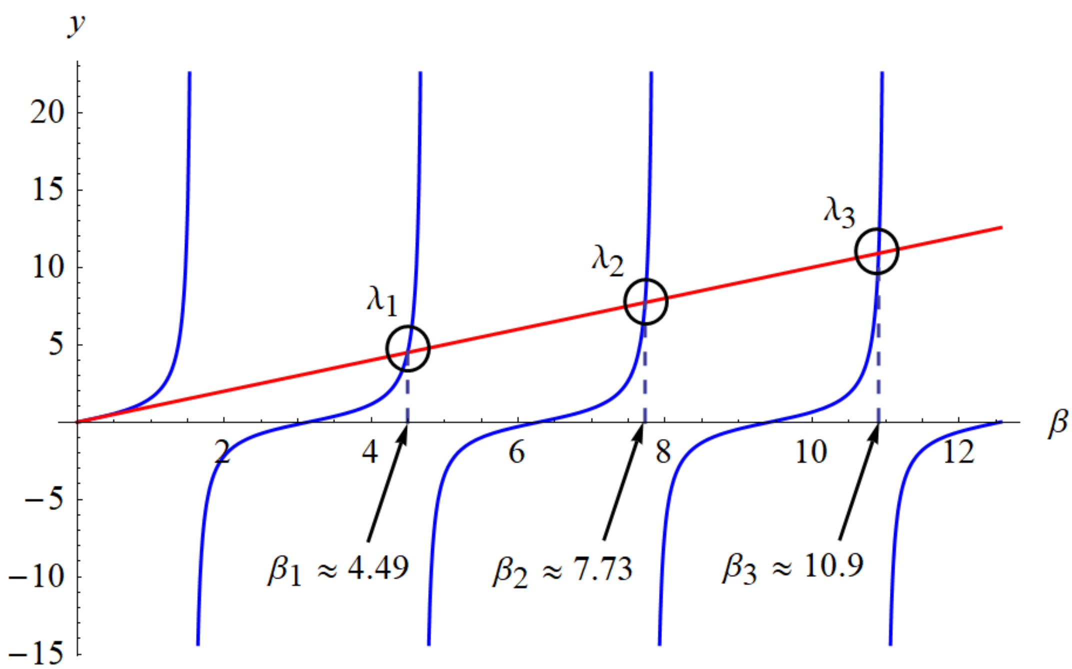

Figure 1: In blue is a plot of $y = \tan\beta$ and in red is a plot of $y = \beta$ for $0 \le \beta \le 4\pi$.

Because the tangent function is periodic, it intersects the linear function an infinite number of times, which means there are an infinite number of positive eigenvalues. The first three intersections are shown here.

Part (d)

Let's find out. If the eigenvalues are negative ($\lambda = -\gamma^2$), then the differential equation becomes

$$X'' = \gamma^2 X.$$

Its general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_5 \cosh \gamma x + C_6 \sinh \gamma x$$

Apply the provided boundary conditions now to determine C_5 and C_6.

$$\begin{aligned} X'(0) + X(0) &= C_5 + C_6\gamma = 0 \\ X(1) &= C_5 \cosh \gamma + C_6 \sinh \gamma = 0 \end{aligned}$$

From the first equation we have $C_5 = -C_6\gamma$. Plug this into the second equation to obtain

$$-C_6\gamma \cosh \gamma + C_6 \sinh \gamma = 0$$

$$C_6 \sinh \gamma = C_6 \gamma \cosh \gamma$$

The negative eigenvalues are given by $\lambda = -\gamma^2$, where γ satisfies the transcendental equation,

$$\tanh \gamma = \gamma.$$

The solutions to $\tanh \gamma = \gamma$ are where the curves, $y = \gamma$ and $y = \tanh \gamma$, intersect.

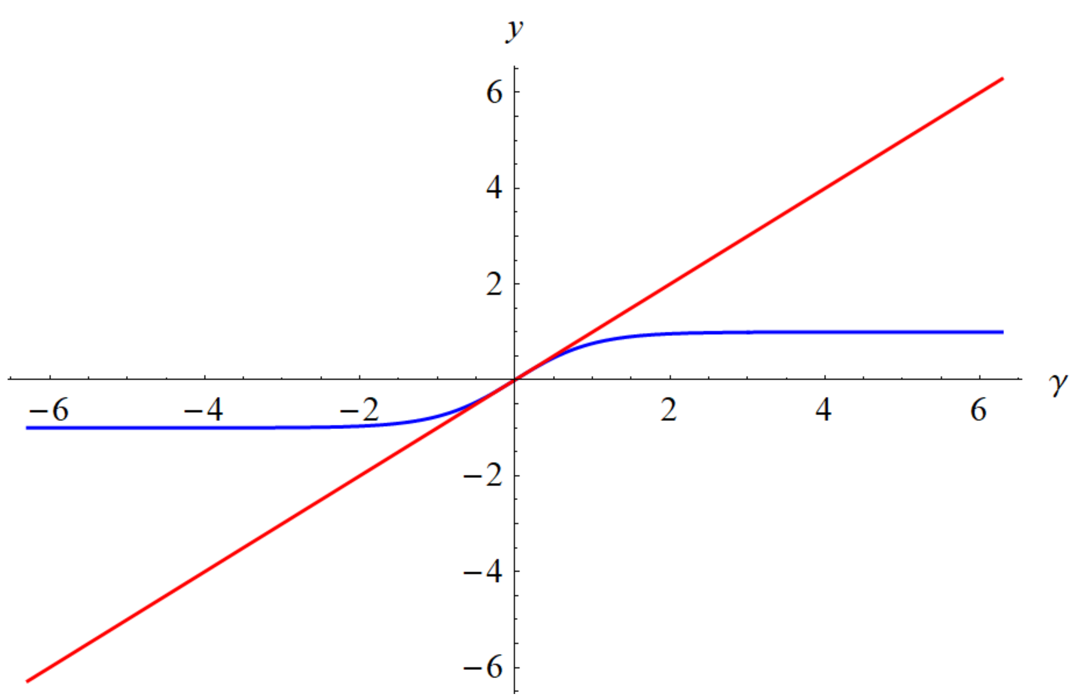

Figure 2: In blue is a plot of $y = \tanh \gamma$ and in red is a plot of $y = \gamma$ for $0 \leq \gamma \leq 2\pi$.

These functions never intersect; therefore, there are no negative eigenvalues. The intersection at $\gamma = 0$ doesn't count since we're looking for nonzero values of λ.