Exercise 1

In the expansion $1 = \sum_{n \text{ odd}} (4/n\pi) \sin nx$, (typo: should be x, not π) valid for $0 < x < \pi$, put $x = \pi/4$ to calculate the sum

$$\left(1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \cdots\right) + \left(\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \frac{1}{15} + \cdots\right) = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots$$

(*Hint:* Since each of the series converges, they can be combined as indicated. However, they cannot be arbitrarily rearranged because they are only conditionally, not absolutely, convergent.)

Solution

If we plug in $x = \pi/4$ to the Fourier sine series for 1, we get

$$\begin{aligned}
1 &= \sum_{n \text{ odd}} (4/n\pi) \sin n\left(\frac{\pi}{4}\right) \\
1 &= \underbrace{\frac{4}{1 * \pi} \sin \frac{\pi}{4}}_{n=1} + \underbrace{\frac{4}{3 * \pi} \sin 3\frac{\pi}{4}}_{n=3} + \underbrace{\frac{4}{5 * \pi} \sin 5\frac{\pi}{4}}_{n=5} + \underbrace{\frac{4}{7 * \pi} \sin 7\frac{\pi}{4}}_{n=7} + \underbrace{\frac{4}{9 * \pi} \sin 9\frac{\pi}{4}}_{n=9} + \cdots \\
1 &= \frac{2\sqrt{2}}{\pi} + \frac{2\sqrt{2}}{3\pi} - \frac{2\sqrt{2}}{5\pi} - \frac{2\sqrt{2}}{7\pi} + \frac{2\sqrt{2}}{9\pi} + \cdots \\
1 &= \frac{2\sqrt{2}}{\pi}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots\right).
\end{aligned}$$

Therefore,

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots \approx 1.111.$$