Exercise 5

Given the Fourier sine series of $\phi(x) \equiv x$ on $(0, l)$. Assume that the series can be integrated term by term, a fact that will be shown later.

(a) Find the Fourier cosine series of the function $x^2/2$. Find the constant of integration that will be the first term in the cosine series.

(b) Then by setting $x = 0$ in your result, find the *sum* of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

Solution

Assume that x has a Fourier sine series expansion with coefficients B_n to be determined.

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = x$$

To solve the equation for B_n, multiply both sides by $\sin(m\pi x/l)$, where m is a positive integer.

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} = x \sin \frac{m\pi x}{l}$$

Now integrate both sides with respect to x over the domain $\phi(x)$ is defined.

$$\int_0^l \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx = \int_0^l x \sin \frac{m\pi x}{l}\, dx$$

Bring the constants in front of the integral.

$$\sum_{n=1}^{\infty} B_n \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx = \int_0^l x \sin \frac{m\pi x}{l}\, dx$$

If $n \neq m$, then the integral on the left is equal to 0 thanks to the orthogonality of the trigonometric functions. This can be verified with the product-to-sum formula for sine. When $n = m$, the integrand becomes $\sin^2(n\pi x/l)$, and the result of the integral is $l/2$.

$$B_n \cdot \frac{l}{2} = \int_0^l x \sin \frac{n\pi x}{l}\, dx$$

Multiply both sides by $2/l$ to solve for B_n.

$$B_n = \frac{2}{l} \int_0^l x \sin \frac{n\pi x}{l}\, dx$$

Use integration by parts to solve the remaining integral.

$$\begin{aligned}
B_n &= \frac{2}{l}\left[-\frac{l}{n\pi}x\cos\frac{n\pi x}{l}\bigg|_0^l - \int_0^l \left(-\frac{l}{n\pi}\right)\cos\frac{n\pi x}{l}\,dx\right] \\
&= \frac{2}{l}\bigg[-\frac{l}{n\pi}(l\cos n\pi - 0) + \underbrace{\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\bigg|_0^l}_{=0}\bigg] \\
&= \frac{2}{l}\left[-\frac{l^2}{n\pi}\cos n\pi\right] \\
&= -\frac{2l}{n\pi}(-1)^n \\
&= \frac{2l}{n\pi}(-1)^{n+1}
\end{aligned}$$

We thus have the Fourier sine series expansion of x.

$$\sum_{n=1}^{\infty}\frac{2l}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{l} = x$$

Part (a)

In order to find the Fourier cosine series of $x^2/2$, integrate both sides with respect to x.

$$\int\sum_{n=1}^{\infty}\frac{2l}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{l}\,dx = \int x\,dx$$

Bring the constants in front of the integral on the left side. Evaluate the integral on the right side.

$$\sum_{n=1}^{\infty}\frac{2l}{n\pi}(-1)^{n+1}\int\sin\frac{n\pi x}{l}\,dx = \frac{x^2}{2} + C$$

Evaluate the integral on the left side.

$$\sum_{n=1}^{\infty}\frac{2l}{n\pi}[-(-1)^n]\cdot\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right) = \frac{x^2}{2} + C$$

$$\sum_{n=1}^{\infty}\frac{2l^2}{n^2\pi^2}(-1)^n\cos\frac{n\pi x}{l} = \frac{x^2}{2} + C \tag{1}$$

This equation holds for all values of x, so the constant C can be determined by setting x to something convenient. Set $x = l$.

$$\sum_{n=1}^{\infty}\frac{2l^2}{n^2\pi^2}(-1)^n\cos n\pi = \frac{l^2}{2} + C$$

Bring the constant in front of the sum and note that $\cos n\pi = (-1)^n$.

$$\frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{l^2}{2} + C$$

This infinite series is known to equal $\pi^2/6$.

$$\frac{2l^2}{\pi^2}\cdot\frac{\pi^2}{6} = \frac{l^2}{2} + C$$

$$\frac{l^2}{3} = \frac{l^2}{2} + C$$

Thus,

$$C = -\frac{l^2}{6}.$$

Substitute this into equation (1).

$$\sum_{n=1}^{\infty} \frac{2l^2}{n^2\pi^2}(-1)^n \cos\frac{n\pi x}{l} = \frac{x^2}{2} - \frac{l^2}{6}$$

Therefore,

$$\frac{l^2}{6} + \sum_{n=1}^{\infty} \frac{2l^2}{n^2\pi^2}(-1)^n \cos\frac{n\pi x}{l} = \frac{x^2}{2}.$$

Part (b)

If we set $x = 0$ in this equation, then

$$\frac{l^2}{6} + \sum_{n=1}^{\infty} \frac{2l^2}{n^2\pi^2}(-1)^n = 0.$$

$$\frac{l^2}{6} + \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = 0$$

$$\frac{l^2}{6} = -\frac{2l^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{\pi^2}{12} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$