Exercise 9

Let $\phi(x)$ be a function of period π. If $\phi(x) = \sum_{n=1}^{\infty} a_n \sin nx$ for all x, find the odd coefficients.

Solution

The fact that $\phi(x)$ has period π means that

$$\phi(x) = \phi(x + \pi). \tag{1}$$

We have

$$\phi(x) = \sum_{n=1}^{\infty} a_n \sin nx,$$

so

$$\begin{aligned}
\phi(x + \pi) &= \sum_{n=1}^{\infty} a_n \sin n(x + \pi) \\
&= \sum_{n=1}^{\infty} a_n \sin(nx + n\pi) \\
&= \sum_{n=1}^{\infty} a_n (\sin nx \cos n\pi + \cos nx \sin n\pi).
\end{aligned}$$

Since n is an integer, $\cos n\pi = (-1)^n$ and $\sin n\pi = 0$.

$$= \sum_{n=1}^{\infty} a_n (-1)^n \sin nx$$

Because of equation (1), the series for $\phi(x)$ must be equal to that for $\phi(x + \pi)$.

$$\phi(x) = \phi(x + \pi) \quad \rightarrow \quad \sum_{n=1}^{\infty} a_n \sin nx = \sum_{n=1}^{\infty} a_n (-1)^n \sin nx$$

For these two series to be equal, their coefficients must be equal.

$$a_n = a_n(-1)^n$$

When n is even, $(-1)^n = 1$, and we have a true statement. However, when n is odd, $(-1)^n = -1$, and the only way the equation can be satisfied is if $a_n = 0$. Therefore, the odd coefficients are all equal to zero.