

Exercise 2

Show that $\cos x + \cos \alpha x$ is periodic if α is a rational number. What is its period?

Solution

A function $\phi(x)$ is said to be periodic if there is a number $p > 0$ such that

$$\phi(x) = \phi(x + p)$$

for all x . $\cos x$ is periodic with period 2π because

$$\begin{aligned}\cos(x + 2\pi) &= \cos x \cos 2\pi - \sin x \sin 2\pi \\ &= \cos x,\end{aligned}$$

and $\cos \alpha x$ is periodic with period $2\pi/\alpha$ since

$$\begin{aligned}\cos \left[\alpha \left(x + \frac{2\pi}{\alpha} \right) \right] &= \cos(\alpha x + 2\pi) \\ &= \cos \alpha x \cos 2\pi - \sin \alpha x \sin 2\pi \\ &= \cos \alpha x.\end{aligned}$$

The sum of $\cos x$ and $\cos \alpha x$ is periodic only if the cosines are in phase at some point. For example, say α is $1/2$. The period of $\cos \alpha x$ is then 4π , twice that of $\cos x$; $\cos x$ has to go through two cycles before it is in phase with $\cos \alpha x$, at which point their sum reaches an absolute maximum.

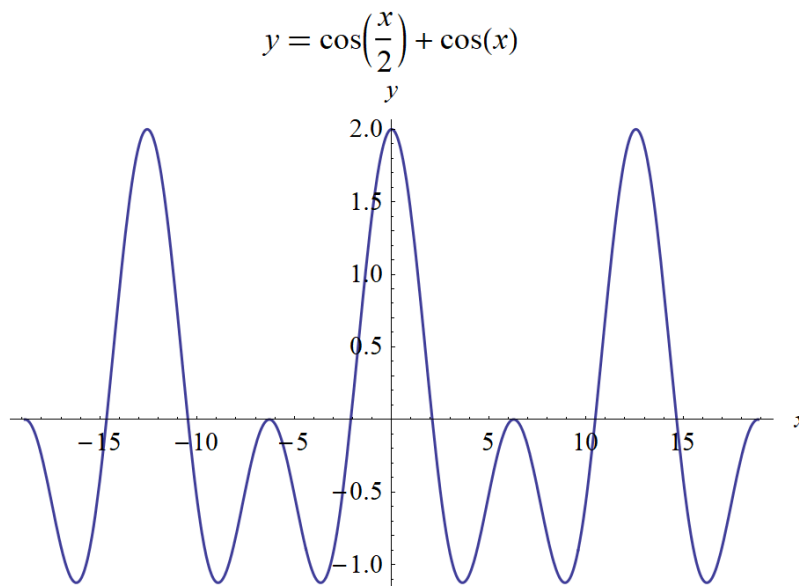


Figure 1: This is a sample graph of $\cos x + \cos \alpha x$ when $\alpha = 1/2$. The period here is $p = 4\pi \approx 12.6$, as can be measured by the distance between absolute maxima.

As another example, consider $\alpha = 2/3$. The period of $\cos \alpha x$ is then 3π . In this case, $\cos \alpha x$ has to complete two cycles and $\cos x$ has to complete three cycles before they are in phase at $p = 6\pi$.

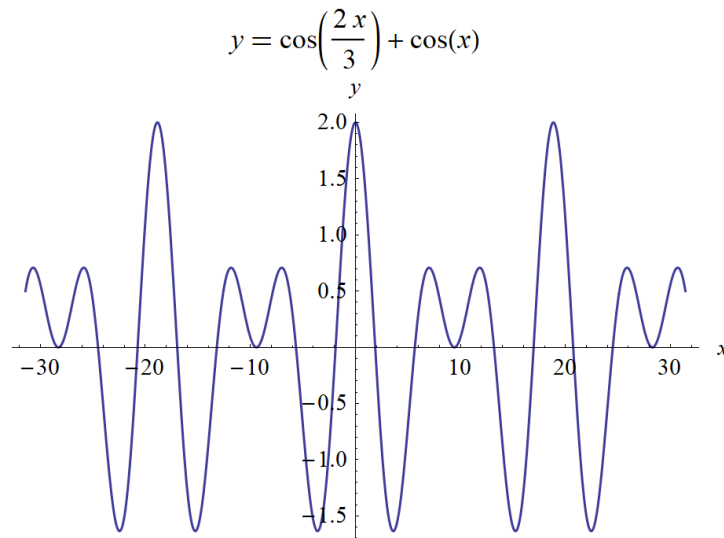


Figure 2: This is a sample graph of $\cos x + \cos \alpha x$ when $\alpha = 2/3$. The period here is $p = 6\pi \approx 18.8$, as can be measured by the distance between absolute maxima.

There is a pattern here: The period of $\cos x + \cos \alpha x$ is π times the least common multiple of $(2/1)$ and $(2/\alpha)$.

$$p = \text{LCM}\left(2, \frac{2}{\alpha}\right) \pi$$

As a test, consider one last example where $\alpha = 8/7$. Our formula predicts that $p = \text{LCM}[2, 2/(8/7)]\pi = 14\pi \approx 43.98$. Finally, note that α must be rational in order for the two cosines to come in phase after a finite number of cycles.

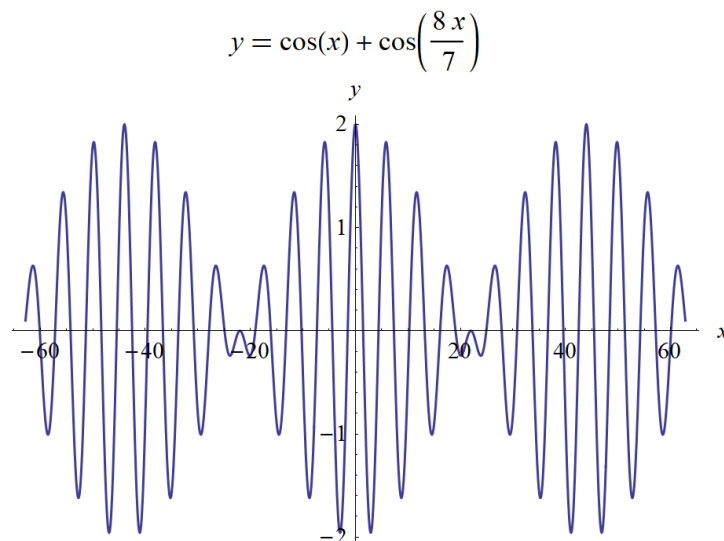


Figure 3: This is a sample graph of $\cos x + \cos \alpha x$ when $\alpha = 8/7$. The period here is $p = 14\pi \approx 43.98$, as can be measured by the distance between absolute maxima. The boxed formula for p is verified.