Exercise 3

Prove property (5) concerning the integrals of even and odd functions.

Solution

The property we have to prove is

$$\int_{-l}^{l} (\text{odd})\, dx = 0 \quad \text{and} \quad \int_{-l}^{l} (\text{even})\, dx = 2\int_{0}^{l} (\text{even})\, dx. \tag{5}$$

The Integral of an Odd Function

Let $f(x)$ denote the odd function: By definition, it satisfies $f(-x) = -f(x)$.

$$\int_{-l}^{l} f(x)\, dx = \int_{-l}^{0} f(x)\, dx + \int_{0}^{l} f(x)\, dx$$

Make the following substitution in the first integral

$$\begin{aligned} u &= -x & &\rightarrow & -u &= x \\ du &= -dx & &\rightarrow & -du &= dx, \end{aligned}$$

and let $u = x$ in the second integral to make the dummy variables the same.

$$\int_{-l}^{l} f(x)\, dx = \int_{l}^{0} f(-u)\,(-du) + \int_{0}^{l} f(u)\, du$$

Use the minus sign to switch the limits of integration.

$$= \int_{0}^{l} f(-u)\, du + \int_{0}^{l} f(u)\, du$$

Use the fact that f is an odd function here.

$$\begin{aligned} &= -\cancel{\int_{0}^{l} f(u)\, du} + \cancel{\int_{0}^{l} f(u)\, du} \\ &= 0 \end{aligned}$$

The Integral of an Even Function

Let $g(x)$ denote the even function: By definition, it satisfies $g(-x) = g(x)$.

$$\int_{-l}^{l} g(x)\, dx = \int_{-l}^{0} g(x)\, dx + \int_{0}^{l} g(x)\, dx$$

Make the following substitution in the first integral

$$\begin{aligned} u &= -x & &\rightarrow & -u &= x \\ du &= -dx & &\rightarrow & -du &= dx, \end{aligned}$$

and let $u = x$ in the second integral to make the dummy variables the same.

$$\int_{-l}^{l} g(x)\,dx = \int_{l}^{0} g(-u)\,(-du) + \int_{0}^{l} g(u)\,du$$

Use the minus sign to switch the limits of integration.

$$= \int_{0}^{l} g(-u)\,du + \int_{0}^{l} g(u)\,du$$

Use the fact that g is an even function here.

$$= \int_{0}^{l} g(u)\,du + \int_{0}^{l} g(u)\,du$$

$$= 2\int_{0}^{l} g(u)\,du$$