Exercise 4

(a) Use (5) to prove that if $\phi(x)$ is an odd function, its full Fourier series on $(-l, l)$ has only sine terms.

(b) Also, if $\phi(x)$ is an even function, its full Fourier series on $(-l, l)$ has only cosine terms. (*Hint:* Don't use the series directly. Use the formulas for the coefficients to show that every second coefficient vanishes.)

Solution

Part (a)

Equation (5) tells us that the integral of an odd function over a symmetric interval is zero and that the integral of an even function over a symmetric interval is twice the integral over the latter half of that interval.

$$\int_{-l}^{l} (\text{odd})\, dx = 0 \quad \text{and} \quad \int_{-l}^{l} (\text{even})\, dx = 2\int_{0}^{l} (\text{even})\, dx. \tag{5}$$

Suppose $\phi(x)$ is an odd function: By definition, it satisfies $\phi(-x) = -\phi(x)$. The full Fourier series expansion of it on $(-l, l)$ is

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}.$$

The formulas for the coefficients are

$$A_n = \frac{1}{l}\int_{-l}^{l} \underbrace{\phi(x)\cos\frac{n\pi x}{l}}_{\text{odd}}\, dx = 0, \quad n = 0, 1, 2, \ldots$$

and

$$B_n = \frac{1}{l}\int_{-l}^{l} \underbrace{\phi(x)\sin\frac{n\pi x}{l}}_{\text{even}}\, dx = \frac{2}{l}\int_{0}^{l} \phi(x)\sin\frac{n\pi x}{l}\, dx, \quad n = 1, 2, \ldots$$

Since $\phi(x)$ is an odd function, $A_n = 0$ $(n = 0, 1, \ldots)$ by the first formula in equation (5). B_n, on the other hand, is not zero. Therefore, only sine terms remain in the full Fourier series. Another way to do the exercise is to substitute $-x$ for x in the Fourier series expansion.

$$\phi(-x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi(-x)}{l} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi(-x)}{l}$$

$$-\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} - \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

Multiply both sides by -1.

$$\phi(x) = -\frac{1}{2}A_0 - \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

Substitute the full Fourier series of $\phi(x)$ on the left.

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} + \cancel{\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}} = -\frac{1}{2}A_0 - \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} + \cancel{\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}}$$

Bring all terms to the left side.

$$A_0 + 2\sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} = 0$$

From this equation we conclude that $A_0 = 0$ and $A_n = 0$. Therefore, if $\phi(x)$ is an odd function, its full Fourier series on $(-l, l)$ has only sine terms.

Part (b)

Suppose now that $\phi(x)$ is an even function: By definition, it satisfies $\phi(-x) = \phi(x)$. The full Fourier series expansion of it on $(-l, l)$ is

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}.$$

The formulas for the coefficients are

$$A_n = \frac{1}{l}\int_{-l}^{l} \underbrace{\phi(x)\cos\frac{n\pi x}{l}}_{\text{even}}\,dx = \frac{2}{l}\int_0^l \phi(x)\cos\frac{n\pi x}{l}\,dx, \quad n = 0, 1, 2, \ldots$$

and

$$B_n = \frac{1}{l}\int_{-l}^{l} \underbrace{\phi(x)\sin\frac{n\pi x}{l}}_{\text{odd}}\,dx = 0, \quad n = 1, 2, \ldots$$

Since $\phi(x)$ is an even function, $B_n = 0$ $(n = 1, 2, \ldots)$ by the first formula in equation (5). A_n, on the other hand, is not zero, which means there are only cosine terms in the Fourier series. Another way to do the exercise is to substitute $-x$ for x in the Fourier series expansion.

$$\begin{aligned}\phi(-x) &= \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi(-x)}{l} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi(-x)}{l} \\ \phi(x) &= \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} - \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\end{aligned}$$

Substitute the full Fourier series of $\phi(x)$ on the left.

$$\cancel{\frac{1}{2}A_0} + \cancel{\sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \cancel{\frac{1}{2}A_0} + \cancel{\sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}} - \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

Bring all terms to the left side.

$$2\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = 0$$

From this equation we conclude that $B_n = 0$. Therefore, if $\phi(x)$ is an even function, its full Fourier series on $(-l, l)$ has only cosine terms.