Exercise 5

Show that the Fourier sine series on $(0, l)$ can be derived from the full Fourier series on $(-l, l)$ as follows. Let $\phi(x)$ be any (continuous) function on $(0, l)$. Let $\tilde{\phi}(x)$ be its odd extension. Write the full series for $\tilde{\phi}(x)$ on $(-l, l)$. [Assume that its sum is $\tilde{\phi}(x)$.] By Exercise 4, this series has only sine terms. Simply restrict your attention to $0 < x < l$ to get the sine series for $\phi(x)$.

Solution

If $\phi(x)$ is a continuous function on $(0, l)$, then it has a Fourier series expansion,

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l},$$

over that interval, where

$$A_n = \frac{2}{l}\int_0^l \phi(x)\cos\frac{n\pi x}{l}\,dx, \quad n = 0, 1, \ldots$$

and

$$B_n = \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx, \quad n = 1, 2, \ldots.$$

Let $\tilde{\phi}$ be the odd extension of $\phi(x)$, that is,

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & 0 < x < l \\ -\phi(-x) & -l < x < 0 \\ 0 & x = 0 \end{cases}.$$

Since it is defined from $(-l, l)$, it has the Fourier series,

$$\tilde{\phi}(x) = \frac{1}{2}\tilde{A}_0 + \sum_{n=1}^{\infty} \tilde{A}_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} \tilde{B}_n \sin\frac{n\pi x}{l},$$

over that interval, where

$$\tilde{A}_n = \frac{1}{l}\int_{-l}^l \underbrace{\tilde{\phi}(x)\cos\frac{n\pi x}{l}}_{\text{odd}}\,dx = 0, \quad n = 0, 1, \ldots$$

and

$$\tilde{B}_n = \frac{1}{l}\int_{-l}^l \underbrace{\tilde{\phi}(x)\sin\frac{n\pi x}{l}}_{\text{even}}\,dx = \frac{2}{l}\int_0^l \tilde{\phi}(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx, \quad n = 1, 2, \ldots.$$

By restricting the domain to $0 < x < l$,

$$\tilde{\phi}(x) = \sum_{n=1}^{\infty} \tilde{B}_n \sin\frac{n\pi x}{l}$$

is equivalent to the Fourier sine series of $\phi(x)$.