Exercise 14

Repeat Exercise 11 for $|x|$.

Solution

The Complex Fourier Series

The complex Fourier series of $|x|$ is

$$|x| = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}. \tag{1}$$

To determine the coefficients c_n, multiply both sides by $e^{-im\pi x/l}$, where m is an integer. We assume that $n \neq m$ for now.

$$|x|e^{-im\pi x/l} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} e^{-im\pi x/l}$$

Integrate both sides with respect to x over the interval $(-l, l)$.

$$\int_{-l}^{l} |x|e^{-im\pi x/l}\,dx = \int_{-l}^{l} \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} e^{-im\pi x/l}\,dx$$

In order to evaluate the integral on the left side, we have to split it over two intervals—one where x is negative and one where x is positive. Bring the constants in front of the integral on the right side.

$$\int_{-l}^{0} (-x)e^{-im\pi x/l}\,dx + \int_{0}^{l} (x)e^{-im\pi x/l}\,dx = \sum_{n=-\infty}^{\infty} c_n \int_{-l}^{l} e^{i(n-m)\pi x/l}\,dx$$

Integration by parts can be used to solve the integrals on the left, but there is a better way: Write the integrands as derivatives with respect to m.

$$\int_{-l}^{0} \left(\frac{l}{i\pi}\frac{\partial}{\partial m}\right) e^{-im\pi x/l}\,dx + \int_{0}^{l} \left(-\frac{l}{i\pi}\frac{\partial}{\partial m}\right) e^{-im\pi x/l}\,dx = \sum_{n=-\infty}^{\infty} c_n \int_{-l}^{l} e^{i(n-m)\pi x/l}\,dx$$

Bring the constants and derivatives in front of the integrals. Integration eliminates the x variable, so total derivatives are used in front of the integrals.

$$\frac{l}{i\pi}\frac{d}{dm}\int_{-l}^{0} e^{-im\pi x/l}\,dx - \frac{l}{i\pi}\frac{d}{dm}\int_{0}^{l} e^{-im\pi x/l}\,dx = \sum_{n=-\infty}^{\infty} c_n \int_{-l}^{l} e^{i(n-m)\pi x/l}\,dx$$

Multiply both sides by $i\pi/l$ and factor the derivative.

$$\frac{d}{dm}\left(\int_{-l}^{0} e^{-im\pi x/l}\,dx - \int_{0}^{l} e^{-im\pi x/l}\,dx\right) = \frac{i\pi}{l}\sum_{n=-\infty}^{\infty} c_n \int_{-l}^{l} e^{i(n-m)\pi x/l}\,dx$$

Evaluate the integrals.

$$\frac{d}{dm}\left(\frac{l}{-im\pi}e^{-im\pi x/l}\bigg|_{-l}^{0}-\frac{l}{-im\pi}e^{-im\pi x/l}\bigg|_{0}^{l}\right)=\frac{i\pi}{l}\sum_{n=-\infty}^{\infty}c_n\cdot\frac{1}{i(n-m)\pi/l}e^{i(n-m)\pi x/l}\bigg|_{-l}^{l}$$

Multiply both sides by $i\pi/l$ again.

$$\frac{d}{dm}\left(-\frac{1}{m}e^{-im\pi x/l}\bigg|_{-l}^{0}+\frac{1}{m}e^{-im\pi x/l}\bigg|_{0}^{l}\right)=-\frac{\pi^2}{l^2}\sum_{n=-\infty}^{\infty}c_n\cdot\frac{1}{i(n-m)\pi/l}e^{i(n-m)\pi x/l}\bigg|_{-l}^{l}$$

Now plug in the limits.

$$\frac{d}{dm}\left(-\frac{1}{m}+\frac{1}{m}e^{im\pi}+\frac{1}{m}e^{-im\pi}-\frac{1}{m}\right)=-\frac{\pi^2}{l^2}\sum_{n=-\infty}^{\infty}c_n\cdot\frac{1}{i(n-m)\pi/l}[e^{i(n-m)\pi}-e^{-i(n-m)\pi}]$$

Simplify the left side.

$$\frac{d}{dm}\left[\frac{1}{m}(e^{im\pi}+e^{-im\pi}-2)\right]=-\frac{\pi^2}{l^2}\sum_{n=-\infty}^{\infty}c_n\cdot\frac{1}{i(n-m)\pi/l}[e^{i(n-m)\pi}-e^{-i(n-m)\pi}]$$

Use Euler's formula to write each exponential function in terms of sine and cosine.

$$\begin{aligned}\frac{d}{dm}&\left[\frac{1}{m}(\cos m\pi+\cancel{i\sin m\pi}+\cos m\pi-\cancel{i\sin m\pi}-2)\right]\\&=-\frac{\pi^2}{l^2}\sum_{n=-\infty}^{\infty}c_n\cdot\frac{1}{i(n-m)\pi/l}\{\cancel{\cos[(n-m)\pi]}+\underbrace{i\sin[(n-m)\pi]}_{=0}\\&\qquad-\cancel{\cos[(n-m)\pi]}+\underbrace{i\sin[(n-m)\pi]}_{=0}\}\end{aligned}$$

We find that every term in the infinite series is zero if $n\neq m$ as a result of the integral. The $n=m$ term is all that remains on the right side.

$$\frac{d}{dn}\left[\frac{1}{n}(2\cos n\pi-2)\right]=-\frac{\pi^2}{l^2}c_n\int_{-l}^{l}e^0\,dx$$

Divide both sides by 2 and evaluate the integral on the right side.

$$\frac{d}{dn}\left(\frac{\cos n\pi-1}{n}\right)=-\frac{\pi^2}{2l^2}c_n\cdot 2l$$

Evaluate the derivative using the quotient rule.

$$\frac{-n\pi\overbrace{\sin n\pi}^{=0}-\overbrace{\cos n\pi}^{=(-1)^n}+1}{n^2}=-\frac{\pi^2}{l}c_n$$

Simplify the left side.

$$\frac{1-(-1)^n}{n^2}=-\frac{\pi^2}{l}c_n$$

Solving for c_n, we have

$$c_n = -\frac{l}{\pi^2}\frac{1-(-1)^n}{n^2} = \begin{cases} 0 & n \text{ even} \\ -\frac{l}{\pi^2}\frac{2}{n^2} & n \text{ odd} \end{cases}.$$

Clearly this formula is valid only when $n \neq 0$. To find c_0, integrate both sides of equation (1) with respect to x over the interval $(-l, l)$.

$$\begin{aligned}
\int_{-l}^{l} |x|\,dx &= \int_{-l}^{l} \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}\,dx \\
\int_{-l}^{0}(-x)\,dx + \int_0^l (x)\,dx &= \int_{-l}^{l}\left(\sum_{n=-\infty}^{-1} c_n e^{in\pi x/l} + c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/l}\right) dx \\
-\int_{-l}^{0} x\,dx + \int_0^l x\,dx &= \sum_{n=-\infty}^{-1} c_n \int_{-l}^{l} e^{in\pi x/l}\,dx + \int_{-l}^{l} c_0\,dx + \sum_{n=1}^{\infty} c_n \int_{-l}^{l} e^{in\pi x/l}\,dx \\
-\frac{x^2}{2}\bigg|_{-l}^{0} + \frac{x^2}{2}\bigg|_0^l &= \sum_{n=-\infty}^{-1} c_n \left.\frac{l}{in\pi} e^{in\pi x/l}\right|_{-l}^{l} + c_0 \cdot 2l + \sum_{n=1}^{\infty} c_n \left.\frac{l}{in\pi} e^{in\pi x/l}\right|_{-l}^{l} \\
\frac{l^2}{2} + \frac{l^2}{2} &= \sum_{n=-\infty}^{-1} c_n \frac{l}{in\pi}(e^{in\pi} - e^{-in\pi}) + c_0 \cdot 2l + \sum_{n=1}^{\infty} c_n \frac{l}{in\pi}(e^{in\pi} - e^{-in\pi}) \\
l^2 &= \sum_{n=-\infty}^{-1} c_n \frac{2l}{n\pi}(\underbrace{\sin n\pi}_{=0}) + c_0 \cdot 2l + \sum_{n=1}^{\infty} c_n \frac{2l}{n\pi}(\underbrace{\sin n\pi}_{=0}) \\
l^2 &= c_0 \cdot 2l
\end{aligned}$$

So then

$$c_0 = \frac{l}{2}.$$

Now that all coefficients have been determined, the Fourier series is known.

$$\begin{aligned}
|x| &= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} \\
&= \sum_{n \text{ even}} c_n e^{in\pi x/l} + \sum_{n=0}^{0} c_n e^{in\pi x/l} + \sum_{n \text{ odd}} c_n e^{in\pi x/l} \\
&= c_0 + \sum_{n=-\infty}^{\infty} c_{2n-1} e^{i(2n-1)\pi x/l} \\
&= \frac{l}{2} + \sum_{n=-\infty}^{\infty}\left(-\frac{l}{\pi^2}\right)\frac{2}{(2n-1)^2} e^{i(2n-1)\pi x/l}
\end{aligned}$$

Therefore, the complex Fourier series of $|x|$ on $(-l, l)$ is

$$|x| = \frac{l}{2} - \frac{2l}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(2n-1)^2} e^{i(2n-1)\pi x/l}.$$

The Real Fourier Series

To obtain the real Fourier series, split up the sum.

$$|x| = \frac{l}{2} - \frac{2l}{\pi^2}\left[\sum_{n=-\infty}^{-1} \frac{1}{(2n-1)^2} e^{i(2n-1)\pi x/l} + e^{-i\pi x/l} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{i(2n-1)\pi x/l}\right]$$

Substitute $n = -k$ in the first sum and $n = k$ in the last sum.

$$= \frac{l}{2} - \frac{2l}{\pi^2}\left[\sum_{-k=-\infty}^{-1} \frac{1}{[2(-k)-1]^2} e^{i[2(-k)-1]\pi x/l} + e^{-i\pi x/l} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} e^{i(2k-1)\pi x/l}\right]$$

k essentially runs from 1 to ∞ in the first sum.

$$= \frac{l}{2} - \frac{2l}{\pi^2}\left[\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} e^{-i(2k+1)\pi x/l} + e^{-i\pi x/l} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} e^{i(2k-1)\pi x/l}\right]$$

The second term in square brackets occurs when $k = 0$ in the first sum.

$$= \frac{l}{2} - \frac{2l}{\pi^2}\left[\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} e^{-i(2k+1)\pi x/l} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} e^{i(2k-1)\pi x/l}\right]$$

Substitute $k = j - 1$ in the first sum and $k = j$ in the second sum.

$$= \frac{l}{2} - \frac{2l}{\pi^2}\left[\sum_{j-1=0}^{\infty} \frac{1}{[2(j-1)+1]^2} e^{-i[2(j-1)+1]\pi x/l} + \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} e^{i(2j-1)\pi x/l}\right]$$

$$= \frac{l}{2} - \frac{2l}{\pi^2}\left[\sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} e^{-i(2j-1)\pi x/l} + \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} e^{i(2j-1)\pi x/l}\right]$$

Combine the two sums.

$$= \frac{l}{2} - \frac{2l}{\pi^2}\sum_{j=1}^{\infty}\left[\frac{1}{(2j-1)^2} e^{-i(2j-1)\pi x/l} + \frac{1}{(2j-1)^2} e^{i(2j-1)\pi x/l}\right]$$

Factor the summand.

$$= \frac{l}{2} - \frac{2l}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{(2j-1)^2}[e^{-i(2j-1)\pi x/l} + e^{i(2j-1)\pi x/l}]$$

$$= \frac{l}{2} - \frac{4l}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{(2j-1)^2}\frac{e^{i(2j-1)\pi x/l} + e^{-i(2j-1)\pi x/l}}{2}$$

$$= \frac{l}{2} - \frac{4l}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{(2j-1)^2}\cos\frac{(2j-1)\pi x}{l}$$

Replacing the dummy index j with n, therefore, the real Fourier series for $|x|$ on $(-l, l)$ is

$$|x| = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{l}.$$

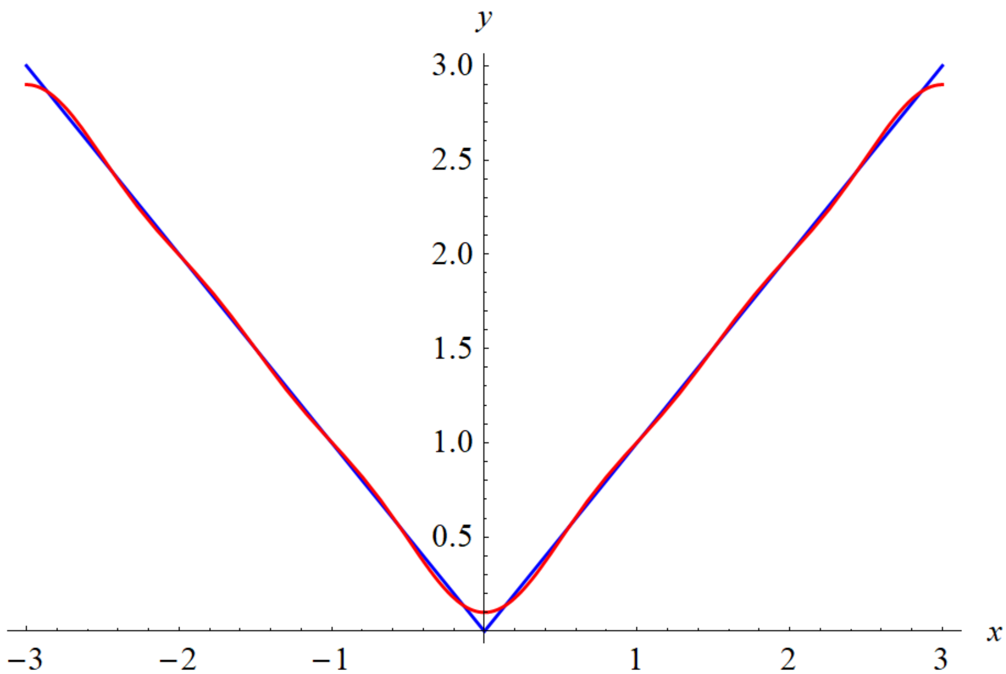

Figure 1: This is a sample plot of $y = |x|$ on $(-3, 3)$ in blue. An approximation to the Fourier series of $|x|$ is plotted in red, where only the first 3 terms in the infinite series have been used.