Exercise 16

Use the De Moivre formulas (11) to derive the standard formulas for $\cos(\theta+\phi)$ and $\sin(\theta+\phi)$.

Solution

The De Moivre formulas are the exponential representations for sine and cosine.

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \quad \text{and} \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \tag{11}$$

The Standard Formula for Cosine

Here we will show that

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

by working backwards.

$$\begin{aligned}
\cos\theta\cos\phi - \sin\theta\sin\phi &= \frac{e^{i\theta}+e^{-i\theta}}{2}\frac{e^{i\phi}+e^{-i\phi}}{2} - \frac{e^{i\theta}-e^{-i\theta}}{2i}\frac{e^{i\phi}-e^{-i\phi}}{2i} \\
&= \frac{1}{4}(e^{i\theta}+e^{-i\theta})(e^{i\phi}+e^{-i\phi}) + \frac{1}{4}(e^{i\theta}-e^{-i\theta})(e^{i\phi}-e^{-i\phi}) \\
&= \frac{1}{4}(e^{i\theta}e^{i\phi} + \cancel{e^{i\theta}e^{-i\phi}} + \cancel{e^{-i\theta}e^{i\phi}} + e^{-i\theta}e^{-i\phi}) \\
&\qquad + \frac{1}{4}(e^{i\theta}e^{i\phi} - \cancel{e^{i\theta}e^{-i\phi}} - \cancel{e^{-i\theta}e^{i\phi}} + e^{-i\theta}e^{-i\phi}) \\
&= \frac{1}{2}e^{i\theta}e^{i\phi} + \frac{1}{2}e^{-i\theta}e^{-i\phi} \\
&= \frac{e^{i(\theta+\phi)}+e^{-i(\theta+\phi)}}{2} \\
&= \cos(\theta+\phi)
\end{aligned}$$

The Standard Formula for Sine

Here we will show that

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

by working backwards as well.

$$\begin{aligned}
\sin\theta\cos\phi + \cos\theta\sin\phi &= \frac{e^{i\theta}-e^{-i\theta}}{2i}\frac{e^{i\phi}+e^{-i\phi}}{2} + \frac{e^{i\theta}+e^{-i\theta}}{2}\frac{e^{i\phi}-e^{-i\phi}}{2i} \\
&= \frac{1}{4i}(e^{i\theta}-e^{-i\theta})(e^{i\phi}+e^{-i\phi}) + \frac{1}{4i}(e^{i\theta}+e^{-i\theta})(e^{i\phi}-e^{-i\phi}) \\
&= \frac{1}{4i}(e^{i\theta}e^{i\phi} + \cancel{e^{i\theta}e^{-i\phi}} - \cancel{e^{-i\theta}e^{i\phi}} - e^{-i\theta}e^{-i\phi}) \\
&\qquad + \frac{1}{4i}(e^{i\theta}e^{i\phi} - \cancel{e^{i\theta}e^{-i\phi}} + \cancel{e^{-i\theta}e^{i\phi}} - e^{-i\theta}e^{-i\phi}) \\
&= \frac{1}{2i}e^{i\theta}e^{i\phi} - \frac{1}{2i}e^{-i\theta}e^{-i\phi} \\
&= \frac{e^{i(\theta+\phi)}-e^{-i(\theta+\phi)}}{2i} \\
&= \sin(\theta+\phi)
\end{aligned}$$