Exercise 4

Consider the problem $u_t = ku_{xx}$ for $0 < x < l$, with the boundary conditions $u(0, t) = U$, $u_x(l, t) = 0$, and the initial condition $u(x, 0) = 0$, where U is a constant.

(a) Find the solution in series form. (*Hint:* Consider $u(x, t) - U$.)

(b) Using a direct argument, show that the series converges for $t > 0$.

(c) If ϵ is a given margin of error, estimate how long a time is required for the value $u(l, t)$ at the endpoint to be approximated by the constant U within error ϵ. (*Hint:* It is an alternating series with first term U, so that the error is less than the next term.)

Solution

Part (a)

Since the boundary conditions are not homogeneous, the method of separation of variables cannot be applied. Make the change of variables,

$$v(x, t) = u(x, t) - U,$$

in order to make them so. Find the derivatives of u in terms of this new variable.

$$\begin{aligned} v_t &= u_t \\ v_{tt} &= u_{tt} \\ v_x &= u_x \\ v_{xx} &= u_{xx} \end{aligned}$$

As a result, v satisfies the same PDE as u.

$$u_t = ku_{xx} \quad \rightarrow \quad v_t = kv_{xx}$$

The initial and boundary conditions associated with it are as follows.

$$\begin{aligned} v(0, t) &= u(0, t) - U = U - U = 0 \\ v_x(l, t) &= u_x(l, t) = 0 \\ v(x, 0) &= u(x, 0) - U = 0 - U = -U \end{aligned}$$

The method of separation of variables can be applied to solve for v because the PDE and its boundary conditions are linear and homogeneous. Assume a product solution of the form $v(x, t) = X(x)T(t)$ and plug it into the PDE

$$v_t = kv_{xx} \quad \rightarrow \quad XT' = kX''T$$

and the boundary conditions.

$$\begin{aligned} v(0, t) &= 0 &\quad \rightarrow \quad X(0)T(t) &= 0 &\quad \rightarrow \quad X(0) &= 0 \\ v_x(l, t) &= 0 &\quad \rightarrow \quad X'(l)T(t) &= 0 &\quad \rightarrow \quad X'(l) &= 0 \end{aligned}$$

Now separate variables in the PDE: divide both sides by kXT to bring all constants and functions of t to the left side and all functions of x to the right side.

$$\frac{T'}{kT} = \frac{X''}{X}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant.

$$\frac{T'}{kT} = \frac{X''}{X} = \lambda$$

Values of λ for which $X(0) = 0$ and $X'(l) = 0$ are satisfied are called the eigenvalues, and the nontrivial functions $X(x)$ associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Assuming λ is positive, the differential equation for X becomes

$$\frac{X''}{X} = \mu^2.$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X'(l) &= \mu(C_1 \sinh \mu l + C_2 \cosh \mu l) = 0 \end{aligned}$$

The second equation reduces to $C_2 \cosh \mu l = 0$. Since hyperbolic cosine is not oscillatory, the only way this equation is satisfied is if $C_2 = 0$. The trivial solution is obtained, so there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Assuming λ is zero, the differential equation for X becomes

$$\frac{X''}{X} = 0.$$

Multiply both sides by X.

$$X'' = 0$$

The general solution is obtained by integrating both sides with respect to x twice. After the first integration, we have

$$X'(x) = C_3.$$

Apply the boundary condition $x'(l) = 0$ to determine C_3.

$$X'(l) = C_3 = 0$$

Integrate both sides of the previous equation with respect to x once more.

$$X(x) = C_4$$

Apply the boundary condition $x(0) = 0$ to determine C_4.

$$X(0) = C_4 = 0$$

The trivial solution $X(x) = 0$ is obtained again, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Assuming λ is negative, the differential equation for X becomes

$$\frac{X''}{X} = -\gamma^2.$$

Multiply both sides by X.

$$X'' = -\gamma^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_5 \cos \gamma x + C_6 \sin \gamma x$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_5 = 0 \\ X'(l) &= \gamma(-C_5 \sin \gamma l + C_6 \cos \gamma l) = 0 \end{aligned}$$

The second equation reduces to

$$\begin{aligned} C_6 \gamma \cos \gamma l &= 0 \\ \cos \gamma l &= 0 \\ \gamma l &= \frac{\pi}{2}(2n-1), \quad n = 1, 2, \ldots \\ \gamma_n &= \frac{\pi}{2l}(2n-1), \quad n = 1, 2, \ldots. \end{aligned}$$

The eigenfunctions associated with the eigenvalues $\lambda = -\gamma^2$ are

$$X(x) = C_6 \sin \mu x \quad \rightarrow \quad X_n(x) = \sin\left[\frac{\pi}{2l}(2n-1)x\right], \quad n = 1, 2, \ldots.$$

Because there are negative eigenvalues, the ODE for T will now be solved.

$$\frac{T'}{kT} = -\gamma^2$$

Multiply both sides by kT.

$$T' = -k\gamma^2 T$$

The general solution is written in terms of the exponential function.

$$T(t) = C_7 e^{-k\gamma^2 t} \quad \rightarrow \quad T_n(t) = \exp\left[-k\frac{\pi^2}{4l^2}(2n-1)^2 t\right], \quad n = 1, 2, \ldots$$

According to the principle of linear superposition, the solution to the PDE for $v(x,t)$ is a linear combination of all products $T_n(t)X_n(x)$ over all the eigenvalues.

$$v(x,t) = \sum_{n=1}^{\infty} B_n \exp\left[-k\frac{\pi^2}{4l^2}(2n-1)^2 t\right] \sin\left[\frac{\pi}{2l}(2n-1)x\right]$$

The initial condition will now be used to determine the coefficients B_n.

$$v(x,0) = \sum_{n=1}^{\infty} B_n \sin\left[\frac{\pi}{2l}(2n-1)x\right] = -U$$

To solve for B_n, multiply both sides by $\sin \gamma_m x$, where m is an integer,

$$\sum_{n=1}^{\infty} B_n \sin\left[\frac{\pi}{2l}(2n-1)x\right] \sin\left[\frac{\pi}{2l}(2m-1)x\right] = -U \sin\left[\frac{\pi}{2l}(2m-1)x\right],$$

and then integrate both sides with respect to x from 0 to l.

$$\begin{aligned}
\int_0^l \sum_{n=1}^{\infty} B_n \sin\left[\frac{\pi}{2l}(2n-1)x\right] \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx &= \int_0^l (-U) \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx \\
\sum_{n=1}^{\infty} B_n \int_0^l \sin\left[\frac{\pi}{2l}(2n-1)x\right] \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx &= U\frac{2l}{\pi(2m-1)} \cos\left[\frac{\pi}{2l}(2m-1)x\right]\Bigg|_0^l
\end{aligned}$$

Because the sine functions are orthogonal, the integral on the left side is zero if $n \neq m$. All the terms in the infinite series vanish as a result except for one, the $n = m$ term.

$$\begin{aligned}
B_n \int_0^l \sin^2\left[\frac{\pi}{2l}(2n-1)x\right] dx &= U\frac{2l}{\pi(2n-1)}(-1) \\
B_n \left(\frac{l}{2}\right) &= -U\frac{2l}{\pi(2n-1)}
\end{aligned}$$

The coefficients are

$$B_n = -U\frac{4}{\pi(2n-1)},$$

which means the solution for v is

$$\begin{aligned}
v(x,t) &= \sum_{n=1}^{\infty}(-U)\frac{4}{\pi(2n-1)} \exp\left[-k\frac{\pi^2}{4l^2}(2n-1)^2 t\right] \sin\left[\frac{\pi}{2l}(2n-1)x\right] \\
&= -U\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1} \exp\left[-k\frac{\pi^2}{4l^2}(2n-1)^2 t\right] \sin\left[\frac{\pi}{2l}(2n-1)x\right].
\end{aligned}$$

Therefore, since $u(x,t) = v(x,t) + U$,

$$u(x,t) = U\left\{1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1} \exp\left[-k\frac{\pi^2}{4l^2}(2n-1)^2 t\right] \sin\left[\frac{\pi}{2l}(2n-1)x\right]\right\}.$$

Part (b)

If we can show that the series in the solution converges absolutely, then it will converge.

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\frac{1}{2n-1}\exp\left[-k\frac{\pi^2}{4l^2}(2n-1)^2t\right]\sin\left[\frac{\pi}{2l}(2n-1)x\right]\right| &= \sum_{n=1}^{\infty}\frac{1}{2n-1}\exp\left[-k\frac{\pi^2}{4l^2}(2n-1)^2t\right]\left|\sin\left[\frac{\pi}{2l}(2n-1)x\right]\right| \\ &\leq \sum_{n=1}^{\infty}\frac{1}{2n-1}\exp\left[-k\frac{\pi^2}{4l^2}(2n-1)^2t\right]\end{aligned}$$

Apply the ratio test to show that this bigger series converges.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{\frac{1}{2n+1}\exp\left[-k\frac{\pi^2}{4l^2}(2n+1)^2t\right]}{\frac{1}{2n-1}\exp\left[-k\frac{\pi^2}{4l^2}(2n-1)^2t\right]}\right| &= \lim_{n\to\infty}\frac{2n-1}{2n+1}\exp\left[k\frac{\pi^2}{4l^2}[(2n-1)^2-(2n+1)^2]t\right] \\ &= \lim_{n\to\infty}\frac{2-\frac{1}{n}}{2+\frac{1}{n}}\exp\left[k\frac{\pi^2}{4l^2}[(2n-1)^2-(2n+1)^2]t\right] \\ &= \lim_{n\to\infty}\exp\left[k\frac{\pi^2}{4l^2}(-8n)t\right] \\ &= \lim_{n\to\infty}\exp\left(-2kn\frac{\pi^2}{l^2}t\right) \\ &= 0\end{aligned}$$

The limit is zero because t is positive. Since the limit is less than one, the bigger series

$$\sum_{n=1}^{\infty}\frac{1}{2n-1}\exp\left[-k\frac{\pi^2}{4l^2}(2n-1)^2t\right]$$

converges by the ratio test. And by the comparison test,

$$\sum_{n=1}^{\infty}\left|\frac{1}{2n-1}\exp\left[-k\frac{\pi^2}{4l^2}(2n-1)^2t\right]\sin\left[\frac{\pi}{2l}(2n-1)x\right]\right|$$

converges as well. Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{2n-1}\exp\left[-k\frac{\pi^2}{4l^2}(2n-1)^2t\right]\sin\left[\frac{\pi}{2l}(2n-1)x\right]$$

converges.

Part (c)

Plug in $x = l$ into the solution.

$$u(l,t) = U - \frac{4U}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\exp\left[-k\frac{\pi^2}{4l^2}(2n-1)^2t\right]\sin\left[\frac{\pi}{2}(2n-1)\right]$$

The sine evaluates to $-(-1)^n$.

$$= U + \frac{4U}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\exp\left[-k\frac{\pi^2}{4l^2}(2n-1)^2t\right]$$

$U - \epsilon$ serves as a lower bound, so set ϵ equal in magnitude to the first term in the infinite series, which is the lowest value in the sequence.

$$\epsilon = \frac{4|U|}{\pi} \exp\left(-k\frac{\pi^2}{4l^2}t\right)$$

Solve this equation for t.

$$\frac{\epsilon\pi}{4|U|} = \exp\left(-k\frac{\pi^2}{4l^2}t\right)$$
$$\ln\frac{\epsilon\pi}{4|U|} = -k\frac{\pi^2}{4l^2}t$$
$$\frac{4l^2}{k\pi^2}\ln\frac{\epsilon\pi}{4|U|} = -t$$

Therefore, the time required to reach $U - \epsilon$ at the endpoint is

$$\begin{aligned} t &= \left|\frac{4l^2}{k\pi^2}\ln\frac{\epsilon\pi}{4|U|}\right| \\ &= \frac{4l^2}{k\pi^2}\left|\ln\frac{\epsilon\pi}{4|U|}\right|. \end{aligned}$$

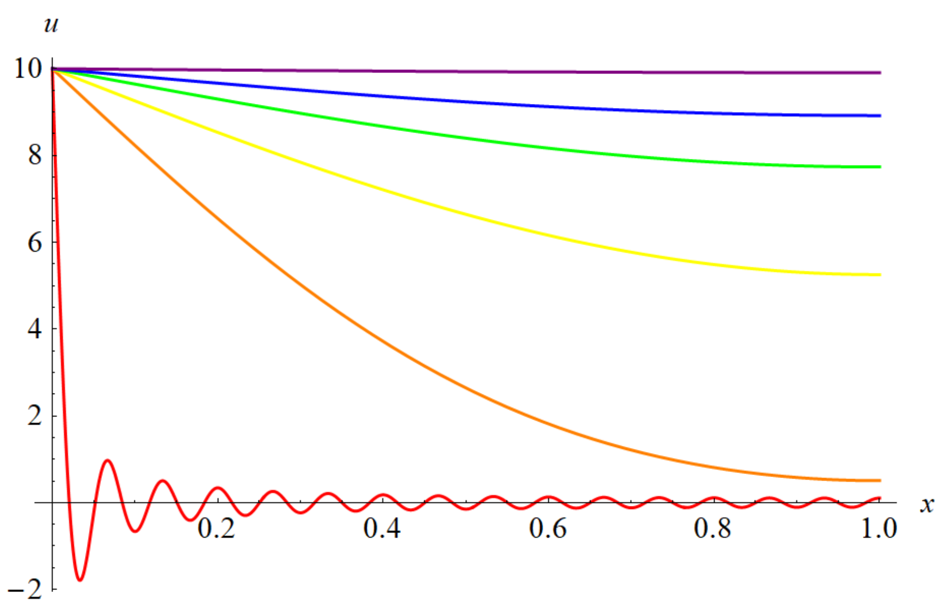

Figure 1: This figure illustrates the solution $u(x,t)$ versus x at various times for $k = 1$, $l = 1$, and $U = 10$ using only the first 30 terms in the infinite series. The curves in red, orange, yellow, green, blue, and purple correspond to the times $t = 0$, $t = 0.1$, $t = 0.4$, $t = 0.7$, $t = 1$, and $t = 2$, respectively.