Exercise 5

(a) Show that the boundary conditions $u(0,t)=0$, $u_x(l,t)=0$ lead to the eigenfunctions $(\sin(\pi x/2l), \sin(3\pi x/2l), \sin(5\pi x/2l), \ldots)$.

(b) If $\phi(x)$ is any function on $(0,l)$, derive the expansion

$$\phi(x) = \sum_{n=0}^{\infty} C_n \sin\left\{\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right\} \quad (0<x<l)$$

by the following method. Extend $\phi(x)$ to the function $\tilde{\phi}$ defined by $\tilde{\phi}(x) = \phi(x)$ for $0 \le x \le l$ and $\tilde{\phi}(x) = \phi(2l-x)$ for $l \le x \le 2l$. (This means that you are extending it *evenly across* $x = l$.) Write the Fourier sine series for $\tilde{\phi}(x)$ on the interval $(0,2l)$ and write the formula for the coefficients.

(c) Show that every second coefficient vanishes.

(d) Rewrite the formula for C_n as an integral of the original function $\phi(x)$ on the interval $(0,l)$.

Solution

Part (a)

Applying the method of separation of variables to either the wave or diffusion equation results in the eigenvalue problem $X'' = \lambda X$ with the boundary conditions, $X(0)=0$ and $X'(l)=0$. Determine if there are any positive eigenvalues by setting $\lambda = \mu^2$ and solving the resulting boundary value problem.

$$X'' = \mu^2 X$$

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Use the boundary conditions to determine C_1 and C_2.

$$X(0) = C_1 = 0$$
$$X'(l) = \mu(C_1 \sinh \mu l + C_2 \cosh \mu l) = 0$$

The second equation reduces to $C_2 \cosh \mu l = 0$. Since hyperbolic cosine is not oscillatory, the only way this equation is satisfied is if $C_2 = 0$. The trivial solution is obtained, so there are no positive eigenvalues. Set $\lambda = 0$ to determine if zero is an eigenvalue.

$$X'' = 0$$

$$X'(x) = C_3.$$

Apply the boundary condition $x'(l) = 0$ to determine C_3.

$$X'(l) = C_3 = 0$$

Integrate both sides of the previous equation with respect to x once more.

$$X(x) = C_4$$

Apply the boundary condition $x(0) = 0$ to determine C_4.

$$X(0) = C_4 = 0$$

The trivial solution $X(x) = 0$ is obtained again, so zero is not an eigenvalue. Determine if there are any negative eigenvalues by setting $\lambda = -\gamma^2$.

$$X'' = -\gamma^2 X$$

$$X(x) = C_5 \cos \gamma x + C_6 \sin \gamma x$$

Use the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_5 = 0 \\ X'(l) &= \gamma(-C_5 \sin \gamma l + C_6 \cos \gamma l) = 0 \end{aligned}$$

The second equation reduces to

$$\begin{aligned} C_6 \gamma \cos \gamma l &= 0 \\ \cos \gamma l &= 0 \\ \gamma l &= \frac{\pi}{2}(2n+1), \quad n = 0, 1, 2, \ldots \\ \gamma_n &= \frac{\pi}{2l}(2n+1), \quad n = 0, 1, 2, \ldots. \end{aligned}$$

The eigenfunctions associated with the eigenvalues $\lambda = -\gamma^2$ are

$$X(x) = C_6 \sin \mu x \quad \rightarrow \quad X_n(x) = \sin \left[\frac{\pi}{2l}(2n+1)x\right], \quad n = 0, 1, 2, \ldots.$$

That is,

$$\begin{aligned} X_0(x) &= \sin \frac{\pi x}{2l} \\ X_1(x) &= \sin \frac{3\pi x}{2l} \\ X_2(x) &= \sin \frac{5\pi x}{2l} \\ &\vdots \end{aligned}.$$

Part (b)

Here we introduce $\tilde{\phi}(x)$, which is defined as

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & \text{for } 0 \leq x \leq l \\ \phi(2l - x) & \text{for } l \leq x \leq 2l \end{cases}.$$

The Fourier sine series for this function over the interval $(0, 2l)$ is

$$\tilde{\phi}(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2l}.$$

To obtain the coefficients B_n, multiply both sides by $\sin(m\pi x/2l)$, where m is an integer,

$$\tilde{\phi}(x)\sin\frac{m\pi x}{2l} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2l}\sin\frac{m\pi x}{2l}$$

and then integrate both sides with respect to x from 0 to $2l$.

$$\begin{aligned}\int_0^{2l}\tilde{\phi}(x)\sin\frac{m\pi x}{2l}\,dx &= \int_0^{2l}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2l}\sin\frac{m\pi x}{2l}\,dx \\ &= \sum_{n=1}^{\infty} B_n \int_0^{2l}\sin\frac{n\pi x}{2l}\sin\frac{m\pi x}{2l}\,dx\end{aligned}$$

Because the sine functions are orthogonal, the integral on the right side is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for one, the $n = m$ term.

$$\begin{aligned}\int_0^{2l}\tilde{\phi}(x)\sin\frac{n\pi x}{2l}\,dx &= B_n\int_0^{2l}\sin^2\frac{n\pi x}{2l}\,dx \\ &= B_n\cdot l\end{aligned}$$

The Fourier coefficients are thus

$$B_n = \frac{1}{l}\int_0^{2l}\tilde{\phi}(x)\sin\frac{n\pi x}{2l}\,dx.$$

Part (c)

Use the definition of $\tilde{\phi}(x)$ to expand the integral.

$$B_n = \frac{1}{l}\left[\int_0^l \phi(x)\sin\frac{n\pi x}{2l}\,dx + \int_l^{2l}\phi(2l-x)\sin\frac{n\pi x}{2l}\,dx\right]$$

Substitute $s = x$ in the first integral and $s = 2l - x$ in the second integral.

$$\begin{aligned}&= \frac{1}{l}\left\{\int_0^l \phi(s)\sin\frac{n\pi s}{2l}\,ds + \int_l^0 \phi(s)\sin\left[\frac{n\pi}{2l}(2l-s)\right](-ds)\right\} \\ &= \frac{1}{l}\left\{\int_0^l \phi(s)\sin\frac{n\pi s}{2l}\,ds + \int_0^l \phi(s)\sin\left[\frac{n\pi}{2l}(2l-s)\right]ds\right\} \\ &= \frac{1}{l}\int_0^l \phi(s)\left\{\sin\frac{n\pi s}{2l} + \sin\left[\frac{n\pi}{2l}(2l-s)\right]\right\}ds\end{aligned}$$

Use the sum-to-product formula for sines.

$$\begin{aligned}&= \frac{1}{l}\int_0^l \phi(s)\left\{2\sin\left[\frac{\frac{n\pi s}{2l} + \frac{n\pi}{2l}(2l-s)}{2}\right]\cos\left[\frac{\frac{n\pi s}{2l} - \frac{n\pi}{2l}(2l-s)}{2}\right]\right\}ds \\ &= \frac{2}{l}\int_0^l \phi(s)\sin\frac{n\pi}{2}\cos\left[\frac{n\pi}{2l}(s-l)\right]ds \\ &= \frac{2}{l}\sin\frac{n\pi}{2}\int_0^l \phi(s)\cos\left[\frac{n\pi}{2l}(l-s)\right]ds\end{aligned}$$

Therefore, because of $\sin(n\pi/2)$, every second coefficient vanishes.

Part (d)

The Fourier sine series for $\tilde{\phi}(x)$ can be simplified (that is, made to converge faster) by summing over the odd values of n only. Make the substitution $n = 2k+1$ in the series

$$\begin{aligned}\tilde{\phi}(x) &= \sum_{2k+1=1}^{\infty} B_{2k+1} \sin\frac{(2k+1)\pi x}{2l} \\ &= \sum_{k=0}^{\infty} B_{2k+1} \sin\left\{\left(k+\frac{1}{2}\right)\frac{\pi x}{l}\right\}\end{aligned}$$

and the corresponding formula for the coefficients.

$$\begin{aligned}B_{2k+1} &= \frac{2}{l}\sin\frac{(2k+1)\pi}{2}\int_0^l \phi(s)\cos\left[\frac{(2k+1)\pi}{2l}(l-s)\right]ds \\ &= \frac{2}{l}\sin\frac{(2k+1)\pi}{2}\int_0^l \phi(s)\left[\cos\frac{(2k+1)\pi}{2}\cos\frac{(2k+1)\pi s}{2l} + \sin\frac{(2k+1)\pi}{2}\sin\frac{(2k+1)\pi s}{2l}\right]ds \\ &= \frac{2}{l}(-1)^k\int_0^l \phi(s)\left[(0)\cos\frac{(2k+1)\pi s}{2l} + (-1)^k\sin\frac{(2k+1)\pi s}{2l}\right]ds \\ &= \frac{2}{l}(-1)^{2k}\int_0^l \phi(s)\sin\frac{(2k+1)\pi s}{2l}\,ds \\ &= \frac{2}{l}\int_0^l \phi(s)\sin\left\{\left(k+\frac{1}{2}\right)\frac{\pi s}{l}\right\}ds\end{aligned}$$

Since k and s are dummy variables, they can be replaced with n and x, respectively. In addition, replace B_{2k+1} with C_n. Then

$$\tilde{\phi}(x) = \sum_{n=0}^{\infty} C_n \sin\left\{\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right\},$$

where

$$C_n = \frac{2}{l}\int_0^l \phi(x)\sin\left\{\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right\}dx.$$

On the interval $(0, l)$, $\tilde{\phi}(x) = \phi(x)$. Therefore,

$$\phi(x) = \sum_{n=0}^{\infty} C_n \sin\left\{\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right\}.$$