Exercise 9

Show that the boundary conditions

$$X(b) = \alpha X(a) + \beta X'(a) \quad \text{and} \quad X'(b) = \gamma X(a) + \delta X'(a)$$

on an interval $a \leq x \leq b$ are symmetric if and only if $\alpha\delta - \beta\gamma = 1$.

Solution

Let $X_1(x)$ and $X_2(x)$ be eigenfunctions that satisfy the boundary conditions at $x = a$ and $x = b$.

$$\begin{aligned} X_1(b) &= \alpha X_1(a) + \beta X_1'(a) & \qquad X_2(b) &= \alpha X_2(a) + \beta X_2'(a) \\ X_1'(b) &= \gamma X_1(a) + \delta X_1'(a) & \qquad X_2'(b) &= \gamma X_2(a) + \delta X_2'(a) \end{aligned}$$

X_1 and X_2 are symmetric if

$$(-X_1'X_2 + X_1X_2')\Big|_a^b = 0.$$

Simplify the left side.

$$\begin{aligned} (-X_1'X_2 + X_1X_2')\Big|_a^b &= -X_1'(b)X_2(b) + X_1(b)X_2'(b) + X_1'(a)X_2(a) - X_1(a)X_2'(a) \\ &= -[\gamma X_1(a) + \delta X_1'(a)][\alpha X_2(a) + \beta X_2'(a)] \\ &\qquad + [\alpha X_1(a) + \beta X_1'(a)][\gamma X_2(a) + \delta X_2'(a)] \\ &\qquad\quad + X_1'(a)X_2(a) - X_1(a)X_2'(a) \\ &= -\cancel{\alpha\gamma X_1(a)X_2(a)} - \beta\gamma X_1(a)X_2'(a) - \alpha\delta X_1'(a)X_2(a) - \cancel{\beta\delta X_1'(a)X_2'(a)} \\ &\qquad + \cancel{\alpha\gamma X_1(a)X_2(a)} + \alpha\delta X_1(a)X_2'(a) + \beta\gamma X_1'(a)X_2(a) + \cancel{\beta\delta X_1'(a)X_2'(a)} \\ &\qquad\quad + X_1'(a)X_2(a) - X_1(a)X_2'(a) \\ &= (-\beta\gamma + \alpha\delta - 1)X_1(a)X_2'(a) + (-\alpha\delta + \beta\gamma + 1)X_1'(a)X_2(a) \end{aligned}$$

The right side is equal to zero if and only if

$$\begin{aligned} -\beta\gamma + \alpha\delta - 1 &= 0 \\ -\alpha\delta + \beta\gamma + 1 &= 0, \end{aligned}$$

or $\alpha\delta - \beta\gamma = 1$.