Exercise 1

(a) Find the real vectors that are orthogonal to the given vectors $[1, 1, 1]$ and $[1, -1, 0]$.

(b) Choosing an answer to (a), expand the vector $[2, -3, 5]$ as a linear combination of these three mutually orthogonal vectors.

Solution

Part (a)

If we take the cross product of the given vectors, then that will give us a vector orthogonal to both of them.

$$[1, 1, 1] \times [1, -1, 0] = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \hat{\mathbf{x}} + \hat{\mathbf{y}} - 2\hat{\mathbf{z}}$$

Any multiple of this vector will be orthogonal to $[1, 1, 1]$ and $[1, -1, 0]$. Therefore,

$$v_\perp = a[1, 1, -2], \quad a \in \mathbb{R}.$$

Part (b)

Choose $[1, 1, -2]$ to be the third vector. Expanding $[2, -3, 5]$ as a linear combination of $[1, 1, 1]$, $[1, -1, 0]$, and $[1, 1, -2]$,

$$[2, -3, 5] = x[1, 1, 1] + y[1, -1, 0] + z[1, 1, -2],$$

we get a system of three equations and three unknowns that we can solve.

$$2 = x + y + z \tag{1}$$

$$-3 = x - y + z \tag{2}$$

$$5 = x - 2z \tag{3}$$

Add equations (1) and (2) together to get

$$-1 = 2x + 2z. \tag{4}$$

Add equations (3) and (4) together to get

$$4 = 3x \quad \rightarrow \quad x = \frac{4}{3}.$$

Plug this result for x back into equation (4) to solve for z.

$$-1 = 2\left(\frac{4}{3}\right) + 2z \quad \rightarrow \quad z = -\frac{11}{6}$$

Now the numbers for x and z can be substituted into either equation (1) or equation (2) to determine y. Equation (2) will be used here.

$$-3 = \frac{4}{3} - y - \frac{11}{6} \quad \rightarrow \quad y = \frac{5}{2}$$

Therefore,

$$[2, -3, 5] = \frac{4}{3}[1, 1, 1] + \frac{5}{2}[1, -1, 0] - \frac{11}{6}[1, 1, -2].$$