Exercise 11

(a) Show that the condition $f(x)f'(x)\big|_a^b \leq 0$ is valid for any function $f(x)$ that satisfies Dirichlet, Neumann, or periodic boundary conditions.

(b) Show that it is also valid for Robin BCs provided that the constants a_0 and a_l are positive.

Solution

Dirichlet Boundary Conditions

A function $f(x)$ that satisfies Dirichlet boundary conditions vanishes at both ends of the interval $a < x < b$.

$$\begin{aligned} f(a) &= 0 \\ f(b) &= 0 \end{aligned}$$

So then

$$\begin{aligned} f(x)f'(x)\big|_a^b &= f(b)f'(b) - f(a)f'(a) \\ &= (0)f'(b) - (0)f'(a) \\ &= 0. \end{aligned}$$

Therefore,

$$f(x)f'(x)\big|_a^b \leq 0$$

is valid for Dirichlet boundary conditions.

Neumann Boundary Conditions

The first derivative of a function $f(x)$ that satisfies Neumann boundary conditions vanishes at both ends of the interval $a < x < b$.

$$\begin{aligned} f'(a) &= 0 \\ f'(b) &= 0 \end{aligned}$$

So then

$$\begin{aligned} f(x)f'(x)\big|_a^b &= f(b)f'(b) - f(a)f'(a) \\ &= f(b)(0) - f(a)(0) \\ &= 0. \end{aligned}$$

Therefore,

$$f(x)f'(x)\big|_a^b \leq 0$$

is valid for Neumann boundary conditions.

Periodic Boundary Conditions

A function with periodic boundary conditions has the same value and slope at both ends of the interval $a < x < b$.

$$\begin{aligned} f(a) &= f(b) \\ f'(a) &= f'(b) \end{aligned}$$

So then

$$\begin{aligned} f(x)f'(x)\big|_a^b &= f(b)f'(b) - f(a)f'(a) \\ &= f(b)f'(b) - f(b)f'(b) \\ &= 0. \end{aligned}$$

Therefore,

$$f(x)f'(x)\big|_a^b \leq 0$$

is valid for periodic boundary conditions.

Robin Boundary Conditions

The first derivative of a function that satisfies Robin boundary conditions is proportional to the value of the function at both ends of the interval $a < x < b$.

$$\begin{aligned} f'(a) &= a_0 f(a) \\ f'(b) &= -a_l f(b) \end{aligned}$$

So then, assuming a_0 and a_l are positive,

$$\begin{aligned} f(x)f'(x)\big|_a^b &= f(b)f'(b) - f(a)f'(a) \\ &= f(b)[-a_l f(b)] - f(a)[a_0 f(a)] \\ &= -a_l[f(b)]^2 - a_0[f(a)]^2 \\ &< 0, \end{aligned}$$

since $[f(a)]^2$ and $[f(b)]^2$ are both positive. Therefore,

$$f(x)f'(x)\big|_a^b \leq 0$$

is valid for Robin boundary conditions in which a_0 and a_l are positive.