Exercise 13

Use Green's first identity to prove Theorem 3. (*Hint:* Substitute $f(x) = X(x) = g(x)$, a real eigenfunction.)

Solution

Theorem 3 reads as follows: Assume the same conditions as in Theorem 1. If

$$f(x)f'(x)\Big|_{x=a}^{x=b} \leq 0 \tag{10}$$

for all (real-valued) functions $f(x)$ satisfying the BCs, then there is *no negative eigenvalue.*

The conditions of Theorem 1 are that we have an eigenvalue problem,

$$-X'' = \lambda X,$$

and that it is subject to symmetric boundary conditions. We will assume equation (10) is true in order to show that there is no negative eigenvalue. From the previous exercise, Green's first identity is

$$\int_a^b f''(x)g(x)\,dx = g(x)f'(x)\Big|_a^b - \int_a^b f'(x)g'(x)\,dx.$$

Following the hint, let $f(x) = X(x)$ and $g(x) = X(x)$ in this formula

$$\int_a^b X''(x)X(x)\,dx = X(x)X'(x)\Big|_a^b - \int_a^b [X'(x)]^2\,dx \tag{1}$$

and in equation (10).

$$X(x)X'(x)\Big|_{x=a}^{x=b} \leq 0 \tag{2}$$

Solve equation (1) for $X(x)X'(x)|_a^b$ and plug it into equation (2).

$$\int_a^b X''(x)X(x)\,dx + \int_a^b [X'(x)]^2\,dx \leq 0$$

From the eigenvalue problem we have $X'' = -\lambda X$, so

$$\int_a^b (-\lambda)[X(x)]^2\,dx + \int_a^b [X'(x)]^2\,dx \leq 0.$$

Bring the constant in front of the integral.

$$(-\lambda)\int_a^b [X(x)]^2\,dx + \int_a^b [X'(x)]^2\,dx \leq 0.$$

Since both integrands are squared, they are positive for all x; thus, the integrals yield positive numbers. λ cannot be negative because otherwise the left side of the inequality would be a positive number, making the inequality we assumed to be true false. Therefore, there is no negative eigenvalue.