Exercise 2

Prove the *Schwarz inequality* (for any pair of functions):

$$|(f,g)| \leq \|f\| \cdot \|g\|.$$

(*Hint:* Consider the expression $\|f+tg\|^2$, where t is a scalar. This expression is a quadratic polynomial of t. Find the value of t where it is a minimum. Play around and the Schwarz inequality will pop out.)

Solution

Consider the expression $\|f+tg\|^2$.

$$\begin{aligned}
\|f+tg\|^2 &= \int_a^b |f(x)+tg(x)|^2\,dx \\
&= \int_a^b [f(x)+tg(x)]\overline{[f(x)+tg(x)]}\,dx \\
&= \int_a^b [f(x)+tg(x)][\overline{f(x)}+\bar{t}\overline{g(x)}]\,dx
\end{aligned}$$

Differentiate both sides with respect to $\bar{t}$, keeping t constant.

$$\begin{aligned}
\frac{\partial}{\partial \bar{t}}\|f+tg\|^2 &= \frac{\partial}{\partial \bar{t}}\int_a^b [f(x)+tg(x)][\overline{f(x)}+\bar{t}\overline{g(x)}]\,dx \\
&= \int_a^b [f(x)+tg(x)]\overline{g(x)}\,dx
\end{aligned}$$

Set the result equal to zero to find the value of t for which the L^2 norm is a minimum.

$$\begin{gathered}
\int_a^b [f(x)+tg(x)]\overline{g(x)}\,dx = 0 \\
\int_a^b [f(x)\overline{g(x)}+tg(x)\overline{g(x)}]\,dx = 0 \\
\int_a^b f(x)\overline{g(x)}\,dx + t\int_a^b g(x)\overline{g(x)}\,dx = 0 \\
(f,g)+t(g,g) = 0 \\
t = -\frac{(f,g)}{(g,g)} = -\frac{(f,g)}{\|g\|^2}
\end{gathered}$$

Now substitute this value of t into the L^2 norm.

$$\begin{aligned}
\left\|f-\frac{(f,g)}{\|g\|^2}g\right\|^2 &= \int_a^b \left|f(x)-\frac{(f,g)}{\|g\|^2}g(x)\right|^2 dx \\
&= \int_a^b \left[f(x)-\frac{(f,g)}{\|g\|^2}g(x)\right]\overline{\left[f(x)-\frac{(f,g)}{\|g\|^2}g(x)\right]}\,dx
\end{aligned}$$

Note that $\|g\|^2$ is a positive real constant, so the complex conjugate does not apply to it.

$$\begin{aligned}
\left\| f - \frac{(f,g)}{\|g\|^2} g \right\|^2 &= \int_a^b \left[f(x) - \frac{(f,g)}{\|g\|^2} g(x) \right] \left[\overline{f(x)} - \frac{\overline{(f,g)}}{\|g\|^2} \overline{g(x)} \right] dx \\
&= \int_a^b \left[f(x)\overline{f(x)} - \frac{\overline{(f,g)}}{\|g\|^2} f(x)\overline{g(x)} - \frac{(f,g)}{\|g\|^2} g(x)\overline{f(x)} + \frac{(f,g)\overline{(f,g)}}{\|g\|^4} g(x)\overline{g(x)} \right] dx \\
&= \int_a^b f(x)\overline{f(x)}\,dx - \frac{\overline{(f,g)}}{\|g\|^2} \int_a^b f(x)\overline{g(x)}\,dx - \frac{(f,g)}{\|g\|^2} \int_a^b g(x)\overline{f(x)}\,dx + \frac{(f,g)\overline{(f,g)}}{\|g\|^4} \int_a^b g(x)\overline{g(x)}\,dx \\
&= (f,f) - \frac{\overline{(f,g)}}{\|g\|^2}(f,g) - \frac{(f,g)}{\|g\|^2}(g,f) + \frac{(f,g)\overline{(f,g)}}{\|g\|^4}(g,g) \\
&= \|f\|^2 - \frac{\overline{(f,g)}}{\|g\|^2}(f,g) - \frac{(f,g)}{\|g\|^2}\overline{(f,g)} + \frac{(f,g)\overline{(f,g)}}{\|g\|^4}\|g\|^2 \\
&= \|f\|^2 - \frac{|(f,g)|^2}{\|g\|^2} - \frac{|(f,g)|^2}{\|g\|^2} + \frac{|(f,g)|^2}{\|g\|^2} \\
&= \|f\|^2 - \frac{|(f,g)|^2}{\|g\|^2}
\end{aligned}$$

Finally, because the L^2 norm is nonnegative,

$$\begin{aligned}
\left\| f - \frac{(f,g)}{\|g\|^2} g \right\|^2 = \|f\|^2 - \frac{|(f,g)|^2}{\|g\|^2} &\geq 0 \\
\|f\|^2\|g\|^2 &\geq |(f,g)|^2.
\end{aligned}$$

Therefore,

$$\|f\|\|g\| \geq |(f,g)|$$

for any two functions, f and g.