Exercise 5

Prove the *Schwarz inequality for infinite series:*

$$\sum a_n b_n \le \left(\sum a_n^2\right)^{1/2} \left(\sum b_n^2\right)^{1/2}.$$

(*Hint:* See the hint in Exercise 2. Prove it first for finite series (ordinary sums) and then pass to the limit.)

Solution

Suppose we have two k-dimensional vectors.

$$\begin{aligned}\mathbf{a} &= \langle a_1, a_2, \ldots, a_k \rangle \\ \mathbf{b} &= \langle b_1, b_2, \ldots, b_k \rangle\end{aligned}$$

Consider the expression $|\mathbf{a} + t\mathbf{b}|^2$.

$$\begin{aligned}|\mathbf{a} + t\mathbf{b}|^2 &= (\mathbf{a} + t\mathbf{b}) \cdot (\mathbf{a} + t\mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot (t\mathbf{b}) + (t\mathbf{b}) \cdot \mathbf{a} + (t\mathbf{b}) \cdot (t\mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} + t\mathbf{a} \cdot \mathbf{b} + t\mathbf{b} \cdot \mathbf{a} + t^2 \mathbf{b} \cdot \mathbf{b} \\ &= \mathbf{a} \cdot \mathbf{a} + t\mathbf{a} \cdot \mathbf{b} + t\mathbf{a} \cdot \mathbf{b} + t^2 \mathbf{b} \cdot \mathbf{b} \\ &= |\mathbf{a}|^2 + 2t\mathbf{a} \cdot \mathbf{b} + t^2 |\mathbf{b}|^2\end{aligned}$$

Differentiate both sides with respect to t.

$$\begin{aligned}\frac{d}{dt}|\mathbf{a} + t\mathbf{b}|^2 &= \frac{d}{dt}(|\mathbf{a}|^2 + 2t\mathbf{a} \cdot \mathbf{b} + t^2|\mathbf{b}|^2) \\ &= 2\mathbf{a} \cdot \mathbf{b} + 2t|\mathbf{b}|^2\end{aligned}$$

Set the result equal to zero to find the value of t that minimizes the norm.

$$2\mathbf{a} \cdot \mathbf{b} + 2t|\mathbf{b}|^2 = 0$$

$$t = -\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}$$

Now substitute this value of t into the norm.

$$\begin{aligned}\left|\mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}\right|^2 &= \left(\mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}\right) \cdot \left(\mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}\right) \\ &= \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \left(-\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}\right) + \left(-\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}\right) \cdot \mathbf{a} + \left(-\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}\right) \cdot \left(-\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}\right) \\ &= \mathbf{a} \cdot \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}(\mathbf{a} \cdot \mathbf{b}) - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}(\mathbf{b} \cdot \mathbf{a}) + \left(-\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\right)^2 \mathbf{b} \cdot \mathbf{b} \\ &= |\mathbf{a}|^2 - \frac{(\mathbf{a} \cdot \mathbf{b})^2}{|\mathbf{b}|^2} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}(\mathbf{a} \cdot \mathbf{b}) + \frac{(\mathbf{a} \cdot \mathbf{b})^2}{|\mathbf{b}|^4}|\mathbf{b}|^2 \\ &= |\mathbf{a}|^2 - \frac{(\mathbf{a} \cdot \mathbf{b})^2}{|\mathbf{b}|^2} - \frac{(\mathbf{a} \cdot \mathbf{b})^2}{|\mathbf{b}|^2} + \frac{(\mathbf{a} \cdot \mathbf{b})^2}{|\mathbf{b}|^2}\end{aligned}$$

Consequently,

$$\left|\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}\right|^2 = |\mathbf{a}|^2 - \frac{(\mathbf{a}\cdot\mathbf{b})^2}{|\mathbf{b}|^2}.$$

Because the norm is nonnegative,

$$\begin{aligned}
\left|\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}\right|^2 = |\mathbf{a}|^2 - \frac{(\mathbf{a}\cdot\mathbf{b})^2}{|\mathbf{b}|^2} &\geq 0 \\
|\mathbf{a}|^2|\mathbf{b}|^2 &\geq (\mathbf{a}\cdot\mathbf{b})^2 \\
|\mathbf{a}||\mathbf{b}| &\geq \mathbf{a}\cdot\mathbf{b} \\
\sqrt{a_1^2 + a_2^2 + \cdots + a_k^2}\sqrt{b_1^2 + b_2^2 + \cdots + b_k^2} &\geq a_1b_1 + a_2b_2 + \cdots + a_kb_k \\
\sqrt{\sum_{n=1}^k a_n^2}\sqrt{\sum_{n=1}^k b_n^2} &\geq \sum_{n=1}^k a_nb_n.
\end{aligned}$$

Therefore, taking the limit of both sides as $k \to \infty$,

$$\sqrt{\sum_{n=1}^\infty a_n^2}\sqrt{\sum_{n=1}^\infty b_n^2} \geq \sum_{n=1}^\infty a_nb_n.$$