

### Exercise 3

Repeat problem (1) for the case of Neumann BCs.

#### Solution

With Neumann boundary conditions, problem (1) on page 147 becomes

$$\begin{aligned}u_t &= ku_{xx} \quad 0 < x < l, \quad t > 0 \\u_x(0, t) &= h(t) \quad u_x(l, t) = j(t) \\u(x, 0) &= 0.\end{aligned}$$

The PDE is the heat equation for a homogeneous one-dimensional rod. The heat flux is prescribed on the boundary at  $x = 0$  and  $x = l$ . Initially the entire rod is at  $0^\circ$ .

#### Method 1 - Using Term-by-Term Differentiation

For this method to work, the boundary conditions must be made homogeneous. Make the change of variables  $v(x, t) = u(x, t) - r(x, t)$ , where  $r(x, t)$  is some function that satisfies the boundary conditions.

$$\begin{aligned}r_x(0, t) &= h(t) \\r_x(l, t) &= j(t)\end{aligned}$$

A suitable function is

$$r(x, t) = \left(x - \frac{x^2}{2l}\right) h(t) + \frac{x^2}{2l} j(t).$$

As a result, the initial and boundary conditions for  $v$  are

$$\begin{aligned}v_x(0, t) &= u_x(0, t) - r_x(0, t) = h(t) - h(t) = 0 \\v_x(l, t) &= u_x(l, t) - r_x(l, t) = j(t) - j(t) = 0 \\v(x, 0) &= u(x, 0) - r(x, 0) = 0 - \left(x - \frac{x^2}{2l}\right) h(0) - \frac{x^2}{2l} j(0) = \left(\frac{x^2}{2l} - x\right) h(0) - \frac{x^2}{2l} j(0).\end{aligned}$$

Substitute  $u(x, t) = v(x, t) + r(x, t)$  into the PDE now to find the one that  $v$  satisfies.

$$\begin{aligned}v_t + r_t &= k(v_{xx} + r_{xx}) \\v_t &= kv_{xx} + kr_{xx} - r_t \\v_t &= kv_{xx} + \frac{k}{l}[j(t) - h(t)] - \left(x - \frac{x^2}{2l}\right) h'(t) - \frac{x^2}{2l} j'(t)\end{aligned}$$

Consequently, the PDE for  $v$  is

$$v_t = kv_{xx} + Q(x, t), \tag{1}$$

where

$$Q(x, t) = \frac{k}{l}[j(t) - h(t)] - \left(x - \frac{x^2}{2l}\right) h'(t) - \frac{x^2}{2l} j'(t).$$

Since it is linear and inhomogeneous, we choose to apply the method of eigenfunction expansion to solve it. Consider the eigenvalue problem of the differential operator involving the spatial variable  $x$

$$\frac{d^2}{dx^2}\phi = \lambda\phi \quad (2)$$

with the same boundary conditions as  $v$ .

$$\begin{aligned}\phi'(0) &= 0 \\ \phi'(l) &= 0\end{aligned}$$

Values of  $\lambda$  for which the boundary conditions are satisfied are known as the eigenvalues, and the nontrivial solutions associated with them are called the eigenfunctions. Equation (2) is known as the one-dimensional Helmholtz equation; the eigenfunctions for it are known to be orthogonal and form a complete set, which will prove useful later.

### Determination of Positive Eigenvalues: $\lambda = \mu^2$

Suppose that  $\lambda$  is positive. Then equation (2) becomes

$$\frac{d^2\phi}{dx^2} = \mu^2\phi.$$

Its solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\begin{aligned}\phi(x) &= C_1 \cosh \mu x + C_2 \sinh \mu x \\ \phi'(x) &= \mu(C_1 \sinh \mu x + C_2 \cosh \mu x)\end{aligned}$$

Apply the boundary conditions to determine  $C_1$  and  $C_2$ .

$$\begin{aligned}\phi'(0) &= \mu C_2 = 0 \\ \phi'(l) &= \mu(C_1 \sinh \mu l + C_2 \cosh \mu l) = 0\end{aligned}$$

Since  $C_2 = 0$ , the second equation reduces to  $\mu C_1 \sinh \mu l = 0$ . Hyperbolic sine is not oscillatory, so the only way this equation is satisfied is if  $C_1 = 0$ . The trivial solution is obtained, so there are no positive eigenvalues.

### Determination of the Zero Eigenvalue: $\lambda = 0$

Suppose that  $\lambda$  is zero. Then equation (2) becomes

$$\frac{d^2\phi}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to  $x$  twice.

$$\phi(x) = C_3 x + C_4$$

Apply the boundary conditions to determine  $C_3$  and  $C_4$ .

$$\begin{aligned}\phi'(0) &= C_3 = 0 \\ \phi'(l) &= C_3 = 0\end{aligned}$$

$C_4$  remains arbitrary, so zero is an eigenvalue, and the eigenfunction associated with it is a constant.

$$\phi(x) = C_4 \quad \rightarrow \quad \phi_0(x) = 1$$

**Determination of Negative Eigenvalues:**  $\lambda = -\gamma^2$

Suppose that  $\lambda$  is negative. Then equation (2) becomes

$$\frac{d^2\phi}{dx^2} = -\gamma^2\phi.$$

Its solution can be written in terms of sine and cosine.

$$\begin{aligned}\phi(x) &= C_5 \cos \gamma x + C_6 \sin \gamma x \\ \phi'(x) &= \gamma(-C_5 \sin \gamma x + C_6 \cos \gamma x)\end{aligned}$$

Apply the boundary conditions to determine  $C_5$  and  $C_6$ .

$$\begin{aligned}\phi'(0) &= \gamma C_6 = 0 \\ \phi'(l) &= \gamma(-C_5 \sin \gamma l + C_6 \cos \gamma l) = 0\end{aligned}$$

Since  $C_6 = 0$ , the second equation reduces to  $-\gamma C_5 \sin \gamma l = 0$ . To avoid getting the trivial solution, we insist that  $C_5 \neq 0$ . Then

$$\begin{aligned}\sin \gamma l &= 0 \\ \gamma l &= n\pi, \quad n = 1, 2, \dots \\ \gamma_n &= \frac{n\pi}{l}, \quad n = 1, 2, \dots\end{aligned}$$

The eigenfunctions associated with these eigenvalues for  $\lambda$  are

$$\phi(x) = C_5 \cos \gamma x \quad \rightarrow \quad \phi_n(x) = \cos \frac{n\pi x}{l}, \quad n = 1, 2, \dots$$

and form a complete set, so the unknown function  $v$  can be expanded in terms of them.

$$v(x, t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos \frac{n\pi x}{l}$$

To determine  $a_0(t)$  and  $a_n(t)$ , the generalized Fourier coefficients, substitute this formula for  $v$  into the PDE it satisfies, equation (1).

$$\begin{aligned}\frac{\partial}{\partial t} \left[ a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos \frac{n\pi x}{l} \right] &= k \frac{\partial^2}{\partial x^2} \left[ a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos \frac{n\pi x}{l} \right] + Q(x, t) \\ \frac{da_0}{dt} + \frac{\partial}{\partial t} \sum_{n=1}^{\infty} a_n(t) \cos \frac{n\pi x}{l} &= k \frac{\partial^2}{\partial x^2} \sum_{n=1}^{\infty} a_n(t) \cos \frac{n\pi x}{l} + Q(x, t)\end{aligned}$$

Because  $v$  satisfies homogeneous boundary conditions and  $v$ ,  $\partial v/\partial x$ , and  $\partial v/\partial t$  are continuous (reasonable assumptions for the temperature profile in a homogeneous solid rod), the two series can in fact be differentiated term by term.

$$\frac{da_0}{dt} + \sum_{n=1}^{\infty} \frac{da_n}{dt} \cos \frac{n\pi x}{l} = k \sum_{n=1}^{\infty} a_n(t) \frac{d^2}{dx^2} \cos \frac{n\pi x}{l} + Q(x, t)$$

The operator applied to the eigenfunction is equal to the eigenvalue times the eigenfunction.

$$\frac{da_0}{dt} + \sum_{n=1}^{\infty} \frac{da_n}{dt} \cos \frac{n\pi x}{l} = k \sum_{n=1}^{\infty} a_n(t) \lambda_n \cos \frac{n\pi x}{l} + Q(x, t)$$

Bring both series to the left side and combine them.

$$\frac{da_0}{dt} + \sum_{n=1}^{\infty} \left[ \frac{da_n}{dt} - ka_n(t) \lambda_n \right] \cos \frac{n\pi x}{l} = Q(x, t) \quad (3)$$

The aim now is to use this Fourier cosine series expansion of  $Q(x, t)$  to obtain ODEs for  $a_0(t)$  and  $a_n(t)$  and then solve for them. In order to get  $da_0/dt$  in equation (3), substitute the function for  $Q(x, t)$  and integrate both sides with respect to  $x$  from 0 to  $l$ .

$$\int_0^l \left\{ \frac{da_0}{dt} + \sum_{n=1}^{\infty} \left[ \frac{da_n}{dt} - ka_n(t) \lambda_n \right] \cos \frac{n\pi x}{l} \right\} dx = \int_0^l \left\{ \frac{k}{l} [j(t) - h(t)] - \left( x - \frac{x^2}{2l} \right) h'(t) - \frac{x^2}{2l} j'(t) \right\} dx$$

Split up the integrals and bring the constants in front of them.

$$\begin{aligned} \frac{da_0}{dt} \int_0^l dx + \sum_{n=1}^{\infty} \left[ \frac{da_n}{dt} - ka_n(t) \lambda_n \right] \underbrace{\int_0^l \cos \frac{n\pi x}{l} dx}_{=0} \\ = \frac{k}{l} [j(t) - h(t)] \int_0^l dx - h'(t) \int_0^l \left( x - \frac{x^2}{2l} \right) dx - j'(t) \int_0^l \frac{x^2}{2l} dx \end{aligned}$$

Evaluate the integrals.

$$\frac{da_0}{dt} \cdot l = \frac{k}{l} [j(t) - h(t)] \cdot l - h'(t) \cdot \frac{l^2}{3} - j'(t) \cdot \frac{l^2}{6}$$

Divide both sides by  $l$  and simplify the right side.

$$\frac{da_0}{dt} = \frac{k}{l} [j(t) - h(t)] - \frac{l}{6} [2h'(t) + j'(t)]$$

Integrate both sides with respect to  $t$  to get  $a_0(t)$ .

$$a_0(t) = \frac{k}{l} \int_0^t [j(s) - h(s)] ds - \frac{l}{6} [2h(t) + j(t)] + C_7 \quad (4)$$

The lower limit of integration is arbitrary as long as  $C_7$  is present. It's chosen here to be 0 because  $t > 0$  in this problem. To solve for the term enclosed in square brackets in equation (3), substitute the function for  $Q(x, t)$  and multiply both sides by  $\cos m\pi x/l$ , where  $m$  is an integer,

$$\begin{aligned} \frac{da_0}{dt} \cos \frac{m\pi x}{l} + \sum_{n=1}^{\infty} \left[ \frac{da_n}{dt} - ka_n(t) \lambda_n \right] \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} \\ = \left\{ \frac{k}{l} [j(t) - h(t)] - \left( x - \frac{x^2}{2l} \right) h'(t) - \frac{x^2}{2l} j'(t) \right\} \cos \frac{m\pi x}{l} \end{aligned}$$

and then integrate both sides with respect to  $x$  from 0 to  $l$ .

$$\begin{aligned} \int_0^l \left\{ \frac{da_0}{dt} \cos \frac{m\pi x}{l} + \sum_{n=1}^{\infty} \left[ \frac{da_n}{dt} - ka_n(t)\lambda_n \right] \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} \right\} dx \\ = \int_0^l \left\{ \frac{k}{l} [j(t) - h(t)] - \left( x - \frac{x^2}{2l} \right) h'(t) - \frac{x^2}{2l} j'(t) \right\} \cos \frac{m\pi x}{l} dx \end{aligned}$$

Split up the integrals and bring the constants in front of them.

$$\begin{aligned} \frac{da_0}{dt} \underbrace{\int_0^l \cos \frac{m\pi x}{l} dx}_{=0} + \sum_{n=1}^{\infty} \left[ \frac{da_n}{dt} - ka_n(t)\lambda_n \right] \int_0^l \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} dx \\ = \frac{k}{l} [j(t) - h(t)] \underbrace{\int_0^l \cos \frac{m\pi x}{l} dx}_{=0} - h'(t) \int_0^l \left( x - \frac{x^2}{2l} \right) \cos \frac{m\pi x}{l} dx - j'(t) \int_0^l \frac{x^2}{2l} \cos \frac{m\pi x}{l} dx \end{aligned}$$

Because the cosine functions are orthogonal, the second integral on the left side is zero if  $n \neq m$ .

As a result, every term in the infinite series vanishes except for one:  $n = m$ .

$$\left[ \frac{da_n}{dt} - ka_n(t)\lambda_n \right] \int_0^l \cos^2 \frac{n\pi x}{l} dx = -h'(t) \int_0^l \left( x - \frac{x^2}{2l} \right) \cos \frac{n\pi x}{l} dx - j'(t) \int_0^l \frac{x^2}{2l} \cos \frac{n\pi x}{l} dx$$

Evaluate the integrals.

$$\left[ \frac{da_n}{dt} - ka_n(t)\lambda_n \right] \cdot \frac{l}{2} = -h'(t) \left( -\frac{l^2}{n^2\pi^2} \right) - j'(t) \left[ \frac{(-1)^n l^2}{n^2\pi^2} \right]$$

Multiply both sides by  $2/l$  and simplify both sides.

$$\frac{da_n}{dt} - k\lambda_n a_n = \frac{2l}{n^2\pi^2} [h'(t) - (-1)^n j'(t)] \tag{5}$$

With the help of the method of eigenfunction expansion, the PDE for  $v$  has been reduced to a first-order inhomogeneous ODE, which can be solved with an integrating factor  $I$ .

$$I = \exp \left[ \int^t (-k\lambda_n) ds \right] = e^{-k\lambda_n t}$$

Multiply both sides of equation (5) by  $I$ .

$$\frac{da_n}{dt} e^{-k\lambda_n t} - k\lambda_n a_n e^{-k\lambda_n t} = \frac{2l}{n^2\pi^2} [h'(t) - (-1)^n j'(t)] e^{-k\lambda_n t}$$

The left side can be written as  $d/dt(Ia_n)$  by the product rule.

$$\frac{d}{dt} (e^{-k\lambda_n t} a_n) = \frac{2l}{n^2\pi^2} [h'(t) - (-1)^n j'(t)] e^{-k\lambda_n t}$$

Integrate both sides with respect to  $t$ .

$$\begin{aligned} e^{-k\lambda_n t} a_n &= \int_0^t \frac{2l}{n^2\pi^2} [h'(s) - (-1)^n j'(s)] e^{-k\lambda_n s} ds + C_8 \\ &= \frac{2l}{n^2\pi^2} \left[ \int_0^t h'(s) e^{-k\lambda_n s} ds - (-1)^n \int_0^t j'(s) e^{-k\lambda_n s} ds \right] + C_8 \end{aligned}$$

Use integration by parts to remove the derivatives from  $h$  and  $j$ .

$$\begin{aligned}
 e^{-k\lambda_n t} a_n &= \frac{2l}{n^2\pi^2} \left\{ h(s)e^{-k\lambda_n s} \Big|_0^t - \int_0^t h(s)(-k\lambda_n)e^{-k\lambda_n s} ds \right. \\
 &\quad \left. - (-1)^n \left[ j(s)e^{-k\lambda_n s} \Big|_0^t - \int_0^t j(s)(-k\lambda_n)e^{-k\lambda_n s} ds \right] \right\} + C_8 \\
 &= \frac{2l}{n^2\pi^2} \left\{ h(t)e^{-k\lambda_n t} - h(0) + k\lambda_n \int_0^t h(s)e^{-k\lambda_n s} ds \right. \\
 &\quad \left. - (-1)^n \left[ j(t)e^{-k\lambda_n t} - j(0) + k\lambda_n \int_0^t j(s)e^{-k\lambda_n s} ds \right] \right\} + C_8 \\
 &= \frac{2l}{n^2\pi^2} \left\{ [h(t) - (-1)^n j(t)]e^{-k\lambda_n t} - [h(0) - (-1)^n j(0)] \right. \\
 &\quad \left. + k\lambda_n \int_0^t [h(s) - (-1)^n j(s)]e^{-k\lambda_n s} ds \right\} + C_8
 \end{aligned}$$

The general solution for  $a_n$  is thus

$$\begin{aligned}
 a_n(t) &= \frac{2le^{k\lambda_n t}}{n^2\pi^2} \left\{ [h(t) - (-1)^n j(t)]e^{-k\lambda_n t} - [h(0) - (-1)^n j(0)] \right. \\
 &\quad \left. + k\lambda_n \int_0^t [h(s) - (-1)^n j(s)]e^{-k\lambda_n s} ds \right\} + C_8 e^{k\lambda_n t}. \quad (6)
 \end{aligned}$$

In order to determine  $C_7$  and  $C_8$ , use the initial condition for  $v$  in combination with the eigenfunction expansion.

$$v(x, 0) = a_0(0) + \sum_{n=1}^{\infty} a_n(0) \cos \frac{n\pi x}{l} = \left( \frac{x^2}{2l} - x \right) h(0) - \frac{x^2}{2l} j(0) \quad (7)$$

This is another Fourier cosine series. To solve for  $a_0(0)$ , integrate both sides with respect to  $x$ .

$$\int_0^l \left[ a_0(0) + \sum_{n=1}^{\infty} a_n(0) \cos \frac{n\pi x}{l} \right] dx = \int_0^l \left[ \left( \frac{x^2}{2l} - x \right) h(0) - \frac{x^2}{2l} j(0) \right] dx$$

Split up the integrals and bring the constants in front of them.

$$a_0(0) \int_0^l dx + \sum_{n=1}^{\infty} a_n(0) \underbrace{\int_0^l \cos \frac{n\pi x}{l} dx}_{=0} = h(0) \int_0^l \left( \frac{x^2}{2l} - x \right) dx - j(0) \int_0^l \frac{x^2}{2l} dx$$

Evaluate the integrals.

$$a_0(0) \cdot l = h(0) \left( -\frac{l^2}{3} \right) - j(0) \left( \frac{l^2}{6} \right)$$

So then

$$a_0(0) = -\frac{l}{6} [2h(0) + j(0)].$$

Set  $t = 0$  in the formula for  $a_0(t)$ , equation (4), and use the initial condition to determine  $C_7$ .

$$a_0(0) = -\frac{l}{6}[2h(0) + j(0)] + C_7 = -\frac{l}{6}[2h(0) + j(0)] \rightarrow C_7 = 0$$

To solve for  $a_n(0)$ , multiply both sides of equation (7) by  $\cos m\pi x/l$

$$a_0(0) \cos \frac{m\pi x}{l} + \sum_{n=1}^{\infty} a_n(0) \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} = \left(\frac{x^2}{2l} - x\right) h(0) \cos \frac{m\pi x}{l} - \frac{x^2}{2l} j(0) \cos \frac{m\pi x}{l}$$

and then integrate both sides from 0 to  $l$ .

$$\int_0^l \left[ a_0(0) \cos \frac{m\pi x}{l} + \sum_{n=1}^{\infty} a_n(0) \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} \right] dx = \int_0^l \left[ \left(\frac{x^2}{2l} - x\right) h(0) \cos \frac{m\pi x}{l} - \frac{x^2}{2l} j(0) \cos \frac{m\pi x}{l} \right] dx$$

Split up the integrals and bring the constants in front of them.

$$a_0(0) \underbrace{\int_0^l \cos \frac{m\pi x}{l} dx}_{=0} + \sum_{n=1}^{\infty} a_n(0) \int_0^l \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} dx = h(0) \int_0^l \left(\frac{x^2}{2l} - x\right) \cos \frac{m\pi x}{l} dx - j(0) \int_0^l \frac{x^2}{2l} \cos \frac{m\pi x}{l} dx$$

Because the cosine functions are orthogonal, the second integral on the left side is zero if  $n \neq m$ . As a result, every term in the infinite series vanishes except for one:  $n = m$ .

$$a_n(0) \int_0^l \cos^2 \frac{n\pi x}{l} dx = h(0) \int_0^l \left(\frac{x^2}{2l} - x\right) \cos \frac{n\pi x}{l} dx - j(0) \int_0^l \frac{x^2}{2l} \cos \frac{n\pi x}{l} dx$$

Evaluate the integrals.

$$a_n(0) \cdot \frac{l}{2} = h(0) \left(\frac{l^2}{n^2\pi^2}\right) - j(0) \left[\frac{(-1)^n l^2}{n^2\pi^2}\right]$$

Solve for  $a_n(0)$ .

$$a_n(0) = \frac{2l}{n^2\pi^2} [h(0) - (-1)^n j(0)]$$

Now set  $t = 0$  in the formula for  $a_n(t)$ , equation (6), and use the initial condition to determine  $C_8$ .

$$a_n(0) = C_8 = \frac{2l}{n^2\pi^2} [h(0) - (-1)^n j(0)]$$

The general solution for  $a_n(t)$  from equation (6) becomes

$$\begin{aligned} a_n(t) &= \frac{2le^{k\lambda_n t}}{n^2\pi^2} \left\{ [h(t) - (-1)^n j(t)]e^{-k\lambda_n t} - \cancel{[h(0) - (-1)^n j(0)]} \right. \\ &\quad \left. + k\lambda_n \int_0^t [h(s) - (-1)^n j(s)]e^{-k\lambda_n s} ds \right\} + \frac{2le^{k\lambda_n t}}{n^2\pi^2} \cancel{[h(0) - (-1)^n j(0)]}. \\ &= \frac{2l}{n^2\pi^2} \left\{ h(t) - (-1)^n j(t) + k\lambda_n \int_0^t [h(s) - (-1)^n j(s)]e^{k\lambda_n(t-s)} ds \right\} \end{aligned}$$

Thus, replacing  $\lambda_n$  with  $-(n\pi/l)^2$ ,

$$a_0(t) = \frac{k}{l} \int_0^t [j(s) - h(s)] ds - \frac{l}{6}[2h(t) + j(t)]$$

$$a_n(t) = \frac{2l}{n^2\pi^2} \left\{ h(t) - (-1)^n j(t) - k \frac{n^2\pi^2}{l^2} \int_0^t [h(s) - (-1)^n j(s)] \exp \left[ -k \frac{n^2\pi^2}{l^2} (t-s) \right] ds \right\}.$$

Now that the generalized Fourier coefficients are known,  $v$  is known as well.

$$\begin{aligned} v(x, t) &= a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos \frac{n\pi x}{l} \\ &= \frac{k}{l} \int_0^t [j(s) - h(s)] ds - \frac{l}{6} [2h(t) + j(t)] \\ &\quad + \sum_{n=1}^{\infty} \frac{2l}{n^2\pi^2} \left\{ h(t) - (-1)^n j(t) - k \frac{n^2\pi^2}{l^2} \int_0^t [h(s) - (-1)^n j(s)] \exp \left[ -k \frac{n^2\pi^2}{l^2} (t-s) \right] ds \right\} \cos \frac{n\pi x}{l} \end{aligned}$$

Since  $u(x, t) = r(x, t) + v(x, t)$ ,

$$\begin{aligned} u(x, t) &= \left( x - \frac{x^2}{2l} \right) h(t) + \frac{x^2}{2l} j(t) + \frac{k}{l} \int_0^t [j(s) - h(s)] ds - \frac{l}{6} [2h(t) + j(t)] \\ &\quad + \sum_{n=1}^{\infty} \frac{2l}{n^2\pi^2} \left\{ h(t) - (-1)^n j(t) - k \frac{n^2\pi^2}{l^2} \int_0^t [h(s) - (-1)^n j(s)] \exp \left[ -k \frac{n^2\pi^2}{l^2} (t-s) \right] ds \right\} \cos \frac{n\pi x}{l}. \end{aligned}$$

Therefore, after combining the first, second, and fourth terms,

$$\begin{aligned} u(x, t) &= \frac{(6lx - 2l^2 - 3x^2)h(t) - (l^2 - 3x^2)j(t)}{6l} + \frac{k}{l} \int_0^t [j(s) - h(s)] ds \\ &\quad - \frac{2l}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left\{ (-1)^n j(t) - h(t) - k \frac{n^2\pi^2}{l^2} \int_0^t [(-1)^n j(s) - h(s)] \exp \left[ -k \frac{n^2\pi^2}{l^2} (t-s) \right] ds \right\} \cos \frac{n\pi x}{l}, \\ &\hspace{25em} 0 \leq x \leq l. \end{aligned}$$



**Method 2 - Mr. Strauss's Way**

The method of eigenfunction expansion will be applied directly to solve for  $u$  as Mr. Strauss does in the textbook, that is, without making the boundary conditions homogeneous. The same eigenvalue problem is considered here as before.

$$\frac{d^2}{dx^2}\phi = \lambda\phi, \quad \begin{aligned} \phi'(0) &= 0 \\ \phi'(l) &= 0 \end{aligned}$$

It was shown that the eigenvalues are zero and  $\lambda_n = -(n\pi/l)^2$  and that the eigenfunctions associated with them are

$$\phi_0(x) = 1 \quad \text{and} \quad \phi_n(x) = \cos \frac{n\pi x}{l}, \quad n = 1, 2, \dots$$

The eigenfunctions of the Helmholtz equation are known to form a complete set, so all of the functions in the PDE can be expanded in terms of them.

$$\begin{aligned} u(x, t) = b_0(t) + \sum_{n=1}^{\infty} b_n(t)\phi_n(x) &\rightarrow \begin{cases} \int_0^l u \, dx = \int_0^l b_0(t) \, dx \\ u\phi_m = b_0\phi_m + \sum_{n=1}^{\infty} b_n\phi_n\phi_m \end{cases} \rightarrow \begin{cases} \int_0^l u \, dx = b_0(t) \cdot l \\ \int_0^l u\phi_n \, dx = b_n \int_0^l \phi_n^2 \, dx \end{cases} \\ \frac{\partial u}{\partial t} = c_0(t) + \sum_{n=1}^{\infty} c_n(t)\phi_n(x) &\rightarrow \begin{cases} \int_0^l \frac{\partial u}{\partial t} \, dx = \int_0^l c_0(t) \, dx \\ \frac{\partial u}{\partial t}\phi_m = c_0\phi_m + \sum_{n=1}^{\infty} c_n\phi_n\phi_m \end{cases} \rightarrow \begin{cases} \int_0^l \frac{\partial u}{\partial t} \, dx = c_0(t) \cdot l \\ \int_0^l \frac{\partial u}{\partial t}\phi_n \, dx = c_n \int_0^l \phi_n^2 \, dx \end{cases} \\ \frac{\partial^2 u}{\partial x^2} = d_0(t) + \sum_{n=1}^{\infty} d_n(t)\phi_n(x) &\rightarrow \begin{cases} \int_0^l \frac{\partial^2 u}{\partial x^2} \, dx = \int_0^l d_0(t) \, dx \\ \frac{\partial^2 u}{\partial x^2}\phi_m = d_0\phi_m + \sum_{n=1}^{\infty} d_n\phi_n\phi_m \end{cases} \rightarrow \begin{cases} \int_0^l \frac{\partial^2 u}{\partial x^2} \, dx = d_0(t) \cdot l \\ \int_0^l \frac{\partial^2 u}{\partial x^2}\phi_n \, dx = d_n \int_0^l \phi_n^2 \, dx \end{cases} \end{aligned}$$

The integral of  $\phi_n^2$  evaluates to  $l/2$ . It should be emphasized that these are generalized Fourier series expansions for  $u$ ,  $\partial u/\partial t$ , and  $\partial^2 u/\partial x^2$ , not product solutions that come about from using the method of separation of variables. Solve the latter equations for the generalized Fourier coefficients.

$$\begin{aligned} \begin{cases} b_0(t) = \frac{1}{l} \int_0^l u \, dx \\ b_n(t) = \frac{2}{l} \int_0^l u\phi_n \, dx \end{cases} &\quad \begin{cases} c_0(t) = \frac{1}{l} \int_0^l \frac{\partial u}{\partial t} \, dx = \frac{d}{dt} \left( \frac{1}{l} \int_0^l u \, dx \right) = \frac{db_0}{dt} \\ c_n(t) = \frac{2}{l} \int_0^l \frac{\partial u}{\partial t} \phi_n \, dx = \frac{d}{dt} \left( \frac{2}{l} \int_0^l u\phi_n \, dx \right) = \frac{db_n}{dt} \end{cases} \\ \begin{cases} d_0(t) = \frac{1}{l} \int_0^l \frac{\partial^2 u}{\partial x^2} \, dx = \frac{1}{l} \frac{\partial u}{\partial x} \Big|_0^l = \frac{1}{l} \left[ \frac{\partial u}{\partial x}(l, t) - \frac{\partial u}{\partial x}(0, t) \right] = \frac{1}{l} [j(t) - h(t)] \\ d_n(t) = \frac{2}{l} \int_0^l \frac{\partial^2 u}{\partial x^2} \phi_n \, dx = \frac{2}{l} \left( \frac{\partial u}{\partial x} \phi_n \Big|_0^l - \int_0^l \frac{\partial u}{\partial x} \frac{d\phi_n}{dx} \, dx \right) \end{cases} \end{aligned}$$

Apply integration by parts once more in order to write  $d_n$  in terms of  $b_n$  and the given boundary

conditions for  $u$ .

$$\begin{aligned}
 d_n(t) &= \frac{2}{l} \left[ j(t) \cos n\pi - h(t) - \int_0^l \frac{\partial u}{\partial x} \left( -\frac{n\pi}{l} \right) \sin \frac{n\pi x}{l} dx \right] \\
 &= \frac{2}{l} \left[ (-1)^n j(s) - h(t) + \frac{n\pi}{l} \int_0^l \frac{\partial u}{\partial x} \sin \frac{n\pi x}{l} dx \right] \\
 &= \frac{2}{l} \left[ (-1)^n j(s) - h(t) + \frac{n\pi}{l} \underbrace{\left( u \sin \frac{n\pi x}{l} \right) \Big|_0^l}_{=0} - \int_0^l u \cdot \frac{n\pi}{l} \cos \frac{n\pi x}{l} dx \right] \\
 &= \frac{2}{l} \left[ (-1)^n j(s) - h(t) - \frac{n^2\pi^2}{l^2} \int_0^l u \cos \frac{n\pi x}{l} dx \right] \\
 &= \frac{2}{l} [(-1)^n j(s) - h(t)] - \frac{n^2\pi^2}{l^2} \cdot \frac{2}{l} \int_0^l u \cos \frac{n\pi x}{l} dx \\
 &= \frac{2}{l} [(-1)^n j(s) - h(t)] - \frac{n^2\pi^2}{l^2} b_n
 \end{aligned}$$

Now that the coefficients are known, substitute the eigenfunction expansions into the PDE.

$$\begin{aligned}
 u_t &= k u_{xx} \\
 c_0(t) + \sum_{n=1}^{\infty} c_n(t) \phi_n(x) &= k \left[ d_0(t) + \sum_{n=1}^{\infty} d_n(t) \phi_n(x) \right] \\
 \frac{db_0}{dt} + \sum_{n=1}^{\infty} \frac{db_n}{dt} \phi_n(x) &= k \left\{ \frac{1}{l} [j(t) - h(t)] + \sum_{n=1}^{\infty} \left\{ \frac{2}{l} [(-1)^n j(s) - h(t)] - \frac{n^2\pi^2}{l^2} b_n \right\} \phi_n(x) \right\} \\
 \frac{db_0}{dt} + \sum_{n=1}^{\infty} \frac{db_n}{dt} \phi_n(x) &= \frac{k}{l} [j(t) - h(t)] + \sum_{n=1}^{\infty} k \left\{ \frac{2}{l} [(-1)^n j(s) - h(t)] - \frac{n^2\pi^2}{l^2} b_n \right\} \phi_n(x)
 \end{aligned}$$

As a result of the method of eigenfunction expansion, the PDE reduces to two ODEs—one for  $b_0(t)$  and one for  $b_n(t)$ .

$$\begin{aligned}
 \frac{db_0}{dt} &= \frac{k}{l} [j(t) - h(t)] \\
 \frac{db_n}{dt} &= k \left\{ \frac{2}{l} [(-1)^n j(s) - h(t)] - \frac{n^2\pi^2}{l^2} b_n \right\}
 \end{aligned}$$

The initial conditions for  $b_0$  and  $b_n$  are obtained from the one for  $u$ .

$$u(x, 0) = b_0(0) + \sum_{n=1}^{\infty} b_n(0) \phi_n(x) = 0 \quad \rightarrow \quad \begin{cases} b_0(0) = 0 \\ b_n(0) = 0 \end{cases}$$

Solve for  $b_0$  by integrating both sides with respect to  $t$ . The lower limit of integration is set to zero to satisfy the initial condition.

$$b_0(t) = \frac{k}{l} \int_0^t [j(s) - h(s)] ds$$

In the ODE for  $b_n$ , distribute  $k$  and bring the term with  $b_n$  to the left side.

$$\frac{db_n}{dt} + k \frac{n^2\pi^2}{l^2} b_n = \frac{2k}{l} [(-1)^n j(s) - h(t)]$$

This first-order inhomogeneous ODE can be solved with an integrating factor  $I$ .

$$I = \exp\left(\int^t k \frac{n^2\pi^2}{l^2} ds\right) = \exp\left(k \frac{n^2\pi^2}{l^2} t\right)$$

Multiply both sides of the previous equation by  $I$ .

$$\frac{db_n}{dt} \exp\left(k \frac{n^2\pi^2}{l^2} t\right) + k \frac{n^2\pi^2}{l^2} b_n \exp\left(k \frac{n^2\pi^2}{l^2} t\right) = \frac{2k}{l} [(-1)^n j(s) - h(t)] \exp\left(k \frac{n^2\pi^2}{l^2} t\right)$$

The left side can be written as  $d/dt(Ib_n)$  by the product rule.

$$\frac{d}{dt} \left[ \exp\left(k \frac{n^2\pi^2}{l^2} t\right) b_n \right] = \frac{2k}{l} [(-1)^n j(s) - h(t)] \exp\left(k \frac{n^2\pi^2}{l^2} t\right)$$

Integrate both sides with respect to  $t$ . The lower limit of integration is set to zero to satisfy the initial condition.

$$\exp\left(k \frac{n^2\pi^2}{l^2} t\right) b_n = \frac{2k}{l} \int_0^t [(-1)^n j(s) - h(s)] \exp\left(k \frac{n^2\pi^2}{l^2} s\right) ds$$

Divide both sides by  $I$  to solve for  $b_n$ .

$$b_n(t) = \frac{2k}{l} \int_0^t [(-1)^n j(s) - h(s)] \exp\left[-k \frac{n^2\pi^2}{l^2} (t - s)\right] ds$$

Now that the generalized Fourier coefficients are known,  $u$  is known as well.

$$\begin{aligned} u(x, t) &= b_0(t) + \sum_{n=1}^{\infty} b_n(t) \phi_n(x) \\ &= \frac{k}{l} \int_0^t [j(s) - h(s)] ds + \sum_{n=1}^{\infty} \left\{ \frac{2k}{l} \int_0^t [(-1)^n j(s) - h(s)] \exp\left[-k \frac{n^2\pi^2}{l^2} (t - s)\right] ds \right\} \cos \frac{n\pi x}{l} \end{aligned}$$

Therefore,

$$u(x, t) = \frac{k}{l} \int_0^t [j(s) - h(s)] ds + \frac{2k}{l} \sum_{n=1}^{\infty} \left\{ \int_0^t [(-1)^n j(s) - h(s)] \exp\left[-k \frac{n^2\pi^2}{l^2} (t - s)\right] ds \right\} \cos \frac{n\pi x}{l},$$

$0 < x < l.$

Note that this solution converges much more slowly than the others, and it does not satisfy the inhomogeneous boundary conditions. These issues are due to the fact that  $\phi'(0) \neq u_x(0, t)$  and  $\phi'(l) \neq u_x(l, t)$ . Making the boundary conditions homogeneous first avoids them. This will be done next in Method 3.

**Method 3 - Without Using Term-by-Term Differentiation**

As in Method 1 the boundary conditions will be made homogeneous by making the substitution  $v(x, t) = u(x, t) - r(x, t)$ , where  $r(x, t)$  is some function that satisfies the boundary conditions. If we choose

$$r(x, t) = \left(x - \frac{x^2}{2l}\right) h(t) + \frac{x^2}{2l} j(t),$$

then the initial boundary value problem that  $v$  was shown to satisfy is

$$\begin{aligned} v_t &= kv_{xx} + Q(x, t), & 0 < x < l, & \quad t > 0 \\ v_x(0, t) &= 0 & v_x(l, t) &= 0 \\ v(x, 0) &= \left(\frac{x^2}{2l} - x\right) h(0) - \frac{x^2}{2l} j(0), \end{aligned}$$

where

$$Q(x, t) = \frac{k}{l}[j(t) - h(t)] - \left(x - \frac{x^2}{2l}\right) h'(t) - \frac{x^2}{2l} j'(t).$$

The same eigenvalue problem is considered here.

$$\frac{d^2}{dx^2} \phi = \lambda \phi, \quad \begin{aligned} \phi'(0) &= 0 \\ \phi'(l) &= 0 \end{aligned}$$

It was shown that the eigenvalues are zero and  $\lambda_n = -(n\pi/l)^2$  and that the eigenfunctions associated with them are

$$\phi_0(x) = 1 \quad \text{and} \quad \phi_n(x) = \cos \frac{n\pi x}{l}, \quad n = 1, 2, \dots$$

The eigenfunctions of the Helmholtz equation are known to form a complete set, so all of the functions in the PDE can be expanded in terms of them.

$$\begin{aligned} v(x, t) = A_0(t) + \sum_{n=1}^{\infty} A_n(t) \phi_n(x) &\rightarrow \begin{cases} \int_0^l v \, dx = \int_0^l A_0(t) \, dx \\ v \phi_m = A_0 \phi_m + \sum_{n=1}^{\infty} A_n \phi_n \phi_m \end{cases} \rightarrow \begin{cases} \int_0^l v \, dx = A_0(t) \cdot l \\ \int_0^l v \phi_n \, dx = A_n \int_0^l \phi_n^2 \, dx \end{cases} \\ \frac{\partial v}{\partial t} = B_0(t) + \sum_{n=1}^{\infty} B_n(t) \phi_n(x) &\rightarrow \begin{cases} \int_0^l \frac{\partial v}{\partial t} \, dx = \int_0^l B_0(t) \, dx \\ \frac{\partial v}{\partial t} \phi_m = B_0 \phi_m + \sum_{n=1}^{\infty} B_n \phi_n \phi_m \end{cases} \rightarrow \begin{cases} \int_0^l \frac{\partial v}{\partial t} \, dx = B_0(t) \cdot l \\ \int_0^l \frac{\partial v}{\partial t} \phi_n \, dx = B_n \int_0^l \phi_n^2 \, dx \end{cases} \\ Q(x, t) = D_0(t) + \sum_{n=1}^{\infty} D_n(t) \phi_n(x) &\rightarrow \begin{cases} \int_0^l Q \, dx = \int_0^l D_0(t) \, dx \\ Q \phi_m = D_0 \phi_m + \sum_{n=1}^{\infty} D_n \phi_n \phi_m \end{cases} \rightarrow \begin{cases} \int_0^l Q \, dx = D_0(t) \cdot l \\ \int_0^l Q \phi_n \, dx = D_n \int_0^l \phi_n^2 \, dx \end{cases} \\ \frac{\partial^2 v}{\partial x^2} = E_0(t) + \sum_{n=1}^{\infty} E_n(t) \phi_n(x) &\rightarrow \begin{cases} \int_0^l \frac{\partial^2 v}{\partial x^2} \, dx = \int_0^l E_0(t) \, dx \\ \frac{\partial^2 v}{\partial x^2} \phi_m = E_0 \phi_m + \sum_{n=1}^{\infty} E_n \phi_n \phi_m \end{cases} \rightarrow \begin{cases} \int_0^l \frac{\partial^2 v}{\partial x^2} \, dx = E_0(t) \cdot l \\ \int_0^l \frac{\partial^2 v}{\partial x^2} \phi_n \, dx = E_n \int_0^l \phi_n^2 \, dx \end{cases} \end{aligned}$$

The integral of  $\phi_n^2$  evaluates to  $l/2$ . It should be emphasized that these are generalized Fourier series expansions for the functions, not product solutions that come about from using the method of separation of variables. Solve the latter equations for the generalized Fourier coefficients.

$$\begin{cases} A_0(t) = \frac{1}{l} \int_0^l v \, dx & B_0(t) = \frac{1}{l} \int_0^l \frac{\partial v}{\partial t} \, dx = \frac{d}{dt} \left( \frac{1}{l} \int_0^l v \, dx \right) = \frac{dA_0}{dt} \\ A_n(t) = \frac{2}{l} \int_0^l v \phi_n \, dx & B_n(t) = \frac{2}{l} \int_0^l \frac{\partial v}{\partial t} \phi_n \, dx = \frac{d}{dt} \left( \frac{2}{l} \int_0^l v \phi_n \, dx \right) = \frac{dA_n}{dt} \\ D_0(t) = \frac{1}{l} \int_0^l Q \, dx = \frac{k}{l} [j(t) - h(t)] - \frac{l}{6} [2h'(t) + j'(t)] \\ D_n(t) = \frac{2}{l} \int_0^l Q \phi_n \, dx = \frac{2l}{n^2 \pi^2} [h'(t) - (-1)^n j'(t)] \\ E_0(t) = \frac{1}{l} \int_0^l \frac{\partial^2 v}{\partial x^2} \, dx = \frac{1}{l} \left. \frac{\partial v}{\partial x} \right|_0^l = \frac{1}{l} \left[ \frac{\partial v}{\partial x}(l, t) - \frac{\partial v}{\partial x}(0, t) \right] = 0 \\ E_n(t) = \frac{2}{l} \int_0^l \frac{\partial^2 v}{\partial x^2} \phi_n \, dx = \frac{2}{l} \left( \left. \frac{\partial v}{\partial x} \phi_n \right|_0^l - \int_0^l \frac{\partial v}{\partial x} \frac{d\phi_n}{dx} \, dx \right) = -\frac{2}{l} \int_0^l \frac{\partial v}{\partial x} \left( -\frac{n\pi}{l} \right) \sin \frac{n\pi x}{l} \, dx \end{cases}$$

Apply integration by parts once more in order to write  $E_n$  in terms of  $A_n$ .

$$\begin{aligned} E_n(t) &= \frac{2n\pi}{l^2} \int_0^l \frac{\partial v}{\partial x} \sin \frac{n\pi x}{l} \, dx \\ &= \frac{2n\pi}{l^2} \left[ \underbrace{v \sin \frac{n\pi x}{l}}_{=0} \Big|_0^l - \int_0^l v \left( \frac{n\pi}{l} \right) \cos \frac{n\pi x}{l} \, dx \right] \\ &= -\frac{n^2 \pi^2}{l^2} \left( \frac{2}{l} \int_0^l v \cos \frac{n\pi x}{l} \, dx \right) \\ &= -\frac{n^2 \pi^2}{l^2} A_n \end{aligned}$$

Now that the coefficients are known, substitute the eigenfunction expansions into the PDE.

$$\begin{aligned} v_t &= kv_{xx} + Q(x, t) \\ B_0(t) + \sum_{n=1}^{\infty} B_n(t) \phi_n(x) &= k \left[ E_0(t) + \sum_{n=1}^{\infty} E_n(t) \phi_n(x) \right] + D_0(t) + \sum_{n=1}^{\infty} D_n(t) \phi_n(x) \\ B_0(t) + \sum_{n=1}^{\infty} B_n(t) \phi_n(x) &= k \sum_{n=1}^{\infty} E_n(t) \phi_n(x) + D_0(t) + \sum_{n=1}^{\infty} D_n(t) \phi_n(x) \\ B_0(t) + \sum_{n=1}^{\infty} B_n(t) \phi_n(x) &= D_0(t) + \sum_{n=1}^{\infty} [D_n(t) + kE_n(t)] \phi_n(x) \end{aligned}$$

As a result of the method of eigenfunction expansion, the PDE reduces to two ODEs—one for  $A_0(t)$  and one for  $A_n(t)$ .

$$\begin{aligned} B_0(t) = D_0(t) &\quad \rightarrow \quad \frac{dA_0}{dt} = \frac{k}{l} [j(t) - h(t)] - \frac{l}{6} [2h'(t) + j'(t)] \\ B_n(t) = D_n(t) + kE_n(t) &\quad \rightarrow \quad \frac{dA_n}{dt} = \frac{2l}{n^2 \pi^2} [h'(t) - (-1)^n j'(t)] - k \frac{n^2 \pi^2}{l^2} A_n \end{aligned}$$

These are the same ODEs that  $a_0(t)$  and  $a_n(t)$  satisfy since  $\lambda_n = -(n\pi/l)^2$ . Because the initial condition for  $v$  is the same as in Method 1,  $A_0(t) = a_0(t)$  and  $A_n(t) = a_n(t)$ .

$$A_0(t) = \frac{k}{l} \int_0^t [j(s) - h(s)] ds - \frac{l}{6} [2h(t) + j(t)]$$

$$A_n(t) = \frac{2l}{n^2\pi^2} \left\{ h(t) - (-1)^n j(t) - k \frac{n^2\pi^2}{l^2} \int_0^t [h(s) - (-1)^n j(s)] \exp \left[ -k \frac{n^2\pi^2}{l^2} (t-s) \right] ds \right\}.$$

Now that the generalized Fourier coefficients are known,  $v$  is known as well.

$$v(x, t) = A_0(t) + \sum_{n=1}^{\infty} A_n(t) \cos \frac{n\pi x}{l}$$

$$= \frac{k}{l} \int_0^t [j(s) - h(s)] ds - \frac{l}{6} [2h(t) + j(t)]$$

$$+ \sum_{n=1}^{\infty} \frac{2l}{n^2\pi^2} \left\{ h(t) - (-1)^n j(t) - k \frac{n^2\pi^2}{l^2} \int_0^t [h(s) - (-1)^n j(s)] \exp \left[ -k \frac{n^2\pi^2}{l^2} (t-s) \right] ds \right\} \cos \frac{n\pi x}{l}$$

Since  $u(x, t) = r(x, t) + v(x, t)$ ,

$$u(x, t) = \left( x - \frac{x^2}{2l} \right) h(t) + \frac{x^2}{2l} j(t) + \frac{k}{l} \int_0^t [j(s) - h(s)] ds - \frac{l}{6} [2h(t) + j(t)]$$

$$+ \sum_{n=1}^{\infty} \frac{2l}{n^2\pi^2} \left\{ h(t) - (-1)^n j(t) - k \frac{n^2\pi^2}{l^2} \int_0^t [h(s) - (-1)^n j(s)] \exp \left[ -k \frac{n^2\pi^2}{l^2} (t-s) \right] ds \right\} \cos \frac{n\pi x}{l}.$$

Therefore, after combining the first, second, and fourth terms,

$$u(x, t) = \frac{(6lx - 2l^2 - 3x^2)h(t) - (l^2 - 3x^2)j(t)}{6l} + \frac{k}{l} \int_0^t [j(s) - h(s)] ds$$

$$- \frac{2l}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left\{ (-1)^n j(t) - h(t) - k \frac{n^2\pi^2}{l^2} \int_0^t [(-1)^n j(s) - h(s)] \exp \left[ -k \frac{n^2\pi^2}{l^2} (t-s) \right] ds \right\} \cos \frac{n\pi x}{l},$$

$$0 \leq x \leq l.$$