Exercise 1

(a) Solve as a series the equation $u_t = u_{xx}$ in $(0,1)$ with $u_x(0,t) = 0$, $u(1,t) = 1$, and $u(x,0) = x^2$. Compute the first two coefficients explicitly.

(b) What is the equilibrium state (the term that does not tend to zero)?

Solution

Because the boundary conditions are inhomogeneous, the method of separation of variables cannot be used to solve the PDE as it is. The fact that the boundary conditions are independent of time implies that there will be a state of equilibrium after a long time has passed. If $u_E(x)$ represents the steady state solution, then we expect $u(x,t)$ to converge to $u_E(x)$ in the limit that $t \to \infty$.

$$\lim_{t\to\infty} u(x,t) = u_E(x)$$

We can solve for $u(x,t)$ by taking advantage of the fact that the PDE is linear. Treat it as the sum of an equilibrium part and a transient part: $u(x,t) = u_E(x) + v(x,t)$. At equilibrium u does not change with time, so it satisfies the time-independent PDE and the boundary conditions at $x = 0$ and $x = 1$.

$$\frac{d^2u_E}{dx^2} = 0, \qquad \frac{du_E}{dx}(0) = 0, \ u_E(1) = 1.$$

Integrate both sides with respect to x.

$$\frac{du_E}{dx} = C_1$$

Apply the first boundary condition here to determine C_1.

$$\frac{du_E}{dx}(0) = C_1 = 0$$

Integrate both sides with respect to x once more.

$$u_E(x) = C_2$$

Apply the second boundary condition here to determine C_2.

$$u_E(1) = C_2 = 1$$

Therefore, we have for the answer to (b)

$$\boxed{u_E(x) = 1.}$$

The aim now is to determine $v(x,t)$. Write the terms of the PDE, u_t and u_{xx}, in terms of it.

$$\begin{aligned} u_t &= v_t \\ u_x &= u_E' + v_x \\ u_{xx} &= u_E'' + v_{xx} = 0 + v_{xx} = v_{xx} \end{aligned}$$

Hence, the PDE for v is the same as for u.

$$v_t = v_{xx}$$

The initial and boundary conditions for it are obtained as follows.

$$\begin{aligned} u(x,0) = u_E(x) + v(x,0) = 1 + v(x,0) = x^2 \quad &\rightarrow \quad v(x,0) = x^2 - 1 \\ u_x(0,t) = u_E'(0) + v_x(0,t) = 0 + v_x(0,t) = 0 \quad &\rightarrow \quad v_x(0,t) = 0 \\ u(1,t) = u_E(1) + v(1,t) = 1 + v(1,t) = 1 \quad &\rightarrow \quad v(1,t) = 0 \end{aligned}$$

Since the PDE and boundary conditions for v are linear and homogeneous, the method of separation of variables can be applied now. Assume a product solution of the form, $v(x,t) = X(x)T(t)$. The PDE becomes

$$v_t = v_{xx} \quad \rightarrow \quad XT' = X''T \quad \rightarrow \quad \frac{T'}{T} = \frac{X''}{X}, \tag{1}$$

and the boundary conditions become

$$\begin{aligned} v_x(0,t) = X'(0)T(t) = 0 \quad &\rightarrow \quad X'(0) = 0 \\ v(1,t) = X(1)T(t) = 0 \quad &\rightarrow \quad X(1) = 0. \end{aligned}$$

Regarding equation (1), the only way a function of t on the left can be equal to a function of x on the right is if both sides are equal to a constant k.

$$\frac{T'}{T} = \frac{X''}{X} = k$$

Values of k for which $X'(0) = 0$ and $X(1) = 0$ are satisfied are called the eigenvalues, and the nontrivial functions $X(x)$ associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $k = \mu^2$

Assuming k is positive, the differential equation for X becomes

$$\frac{X''}{X} = \mu^2.$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_3 \cosh \mu x + C_4 \sinh \mu x$$

Now use the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X'(0) &= \mu C_4 = 0 \\ X(1) &= C_3 \cosh \mu + C_4 \sinh \mu = 0 \end{aligned}$$

We see that $C_3 = 0$ and $C_4 = 0$. Hence, only the trivial solution $X(x) = 0$ results from considering positive values for k, and there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $k = 0$

Assuming k is zero, the differential equation for X becomes

$$\frac{X''}{X} = 0.$$

Multiply both sides by X.

$$X'' = 0$$

The general solution is a linear function.

$$X(x) = C_5 x + C_6$$

Now use the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} X'(0) &= C_5 = 0 \\ X(1) &= C_5 + C_6 = 0 \end{aligned}$$

We see that $C_5 = 0$ and $C_6 = 0$. Hence, only the trivial solution $X(x) = 0$ results from considering $k = 0$, and zero is not an eigenvalue.

Determination of Negative Eigenvalues: $k = -\lambda^2$

Assuming k is negative, the differential equation for X becomes

$$\frac{X''}{X} = -\lambda^2.$$

Multiply both sides by X.

$$X'' = -\lambda^2 X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_7 \cos \lambda x + C_8 \sin \lambda x$$

Now use the boundary conditions to determine C_7 and C_8.

$$\begin{aligned} X'(0) &= \lambda C_8 = 0 \\ X(1) &= C_7 \cos \lambda + C_8 \sin \lambda = 0 \end{aligned}$$

The first equation implies $C_8 = 0$, so the second equation simplifies to $C_7 \cos \lambda = 0$. To avoid getting the trivial solution, we insist that $C_7 \neq 0$. Doing so yields an equation for the eigenvalues.

$$\cos \lambda = 0$$

Solve for λ.

$$\lambda = \lambda_n = \frac{1}{2}(2n+1)\pi, \quad n = 0, 1, \ldots$$

The eigenfunctions associated with these eigenvalues are

$$X(x) = C_7 \cos \lambda x \quad \to \quad X_n(x) = \cos \lambda_n x, \quad n = 0, 1, \ldots.$$

Now solve the differential equation for $T(t)$.

$$\frac{T'}{T} = -\lambda^2$$

The left side is just the derivative of $\ln T$.

$$\frac{d}{dt}(\ln T) = -\lambda^2$$

Integrate both sides with respect to t.

$$\ln T = -\lambda^2 t + C_9$$

Exponentiate both sides.

$$T(t) = e^{-\lambda^2 t + C_9} = e^{-\lambda^2 t} e^{C_9}$$

Use a new constant of integration.

$$T(t) = C_{10} e^{-\lambda^2 t} \quad \rightarrow \quad T_n(t) = e^{-\lambda_n^2 t}$$

According to the principle of linear superposition, the solution to the PDE for $v(x,t)$ is a linear combination of all products $T_n(t)X_n(x)$ over all the eigenvalues.

$$\begin{aligned} v(x,t) &= \sum_{n=0}^{\infty} A_n T_n(t) X_n(x) \\ &= \sum_{n=0}^{\infty} A_n e^{-\lambda_n^2 t} \cos \lambda_n x \\ &= \sum_{n=0}^{\infty} A_n e^{-\frac{1}{4}(2n+1)^2 \pi^2 t} \cos\left[\frac{\pi}{2}(2n+1)x\right] \end{aligned}$$

The final task is to use Fourier's method and the initial condition to determine A_n.

$$v(x,0) = \sum_{n=0}^{\infty} A_n \cos\left[\frac{\pi}{2}(2n+1)x\right]$$

Multiply both sides by $\cos \lambda_m x$, where m is an integer.

$$v(x,0)\cos\left[\frac{\pi}{2}(2m+1)x\right] = \sum_{n=0}^{\infty} A_n \cos\left[\frac{\pi}{2}(2n+1)x\right] \cos\left[\frac{\pi}{2}(2m+1)x\right]$$

Integrate both sides with respect to x over the domain of interest $(0,1)$.

$$\int_0^1 v(x,0)\cos\left[\frac{\pi}{2}(2m+1)x\right] dx = \int_0^1 \sum_{n=0}^{\infty} A_n \cos\left[\frac{\pi}{2}(2n+1)x\right] \cos\left[\frac{\pi}{2}(2m+1)x\right] dx$$

Substitute $v(x,0) = x^2 - 1$ and bring the integral inside the sum on the right side.

$$\int_0^1 (x^2-1)\cos\left[\frac{\pi}{2}(2m+1)x\right] dx = \sum_{n=0}^{\infty} A_n \underbrace{\int_0^1 \cos\left[\frac{\pi}{2}(2n+1)x\right] \cos\left[\frac{\pi}{2}(2m+1)x\right] dx}_{=\frac{1}{2}\delta_{nm}}$$

It is thanks to the orthogonality of the eigenfunctions that the integral on the right side vanishes for every value of n except for $n = m$, where it evaluates to $1/2$. This can be verified with the product-to-sum formula for cosine.

$$\frac{1}{2}A_n = \int_0^1 (x^2 - 1)\cos\left[\frac{\pi}{2}(2n+1)x\right] dx$$

Evaluate the remaining integral by splitting it apart.

$$\begin{aligned}
\frac{1}{2}A_n &= \int_0^1 x^2 \cos\lambda_n x\, dx - \int_0^1 \cos\lambda_n x\, dx \\
&= \left(\frac{2}{\lambda_n^2} x\cos\lambda_n x + \frac{\lambda_n^2 x^2 - 2}{\lambda_n^3}\sin\lambda_n x\right)\Bigg|_0^1 - \frac{1}{\lambda_n}\sin\lambda_n x\Bigg|_0^1 \\
&= \left(\frac{2}{\lambda_n^2}\cos\lambda_n + \frac{\lambda_n^2 - 2}{\lambda_n^3}\sin\lambda_n\right) - \frac{1}{\lambda_n}\sin\lambda_n \\
&= \frac{2}{\lambda_n^2}\underbrace{\cos\lambda_n}_{=0} + \cancel{\frac{1}{\lambda_n}\sin\lambda_n} - \frac{2}{\lambda_n^3}\sin\lambda_n - \cancel{\frac{1}{\lambda_n}\sin\lambda_n} \\
&= -\frac{2}{\lambda_n^3}\sin\lambda_n \\
&= -\frac{2}{\lambda_n^3}(-1)^n \\
&= \frac{16(-1)^{n+1}}{\pi^3(2n+1)^3}
\end{aligned}$$

So the coefficient is

$$A_n = \frac{32(-1)^{n+1}}{\pi^3(2n+1)^3}.$$

$v(x,t)$ is finally known.

$$v(x,t) = \sum_{n=0}^{\infty} \frac{32(-1)^{n+1}}{\pi^3(2n+1)^3} e^{-\frac{1}{4}(2n+1)^2\pi^2 t} \cos\left[\frac{\pi}{2}(2n+1)x\right]$$

Therefore,

$$\begin{aligned}
u(x,t) &= 1 + \frac{32}{\pi^3}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)^3} e^{-\frac{1}{4}(2n+1)^2\pi^2 t}\cos\left[\frac{\pi}{2}(2n+1)x\right] \\
&= 1 - \frac{32}{\pi^3} e^{-\frac{\pi^2}{4}t}\cos\left(\frac{\pi}{2}x\right) + \frac{32}{27\pi^3} e^{-\frac{9\pi^2}{4}t}\cos\left(\frac{3\pi}{2}x\right) - \cdots .
\end{aligned}$$

Again, the equilibrium solution is $u_E(x) = 1$.

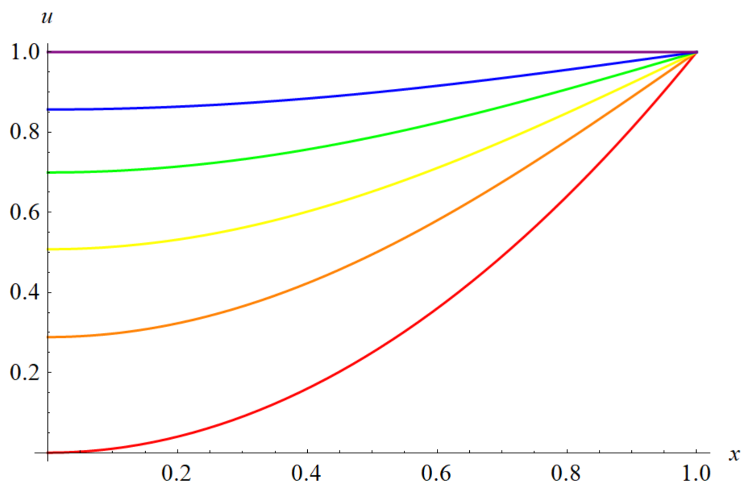

Figure 1: This is a plot of the solution u as a function of x for various times. The curves in red, orange, yellow, green, blue, and purple correspond to $t = 0$, $t = 0.15$, $t = 0.3$, $t = 0.5$, $t = 0.8$, and $t = 10$, respectively. Note that as $t \to \infty$ the graph approaches the equilibrium solution $u_E(x) = 1$.

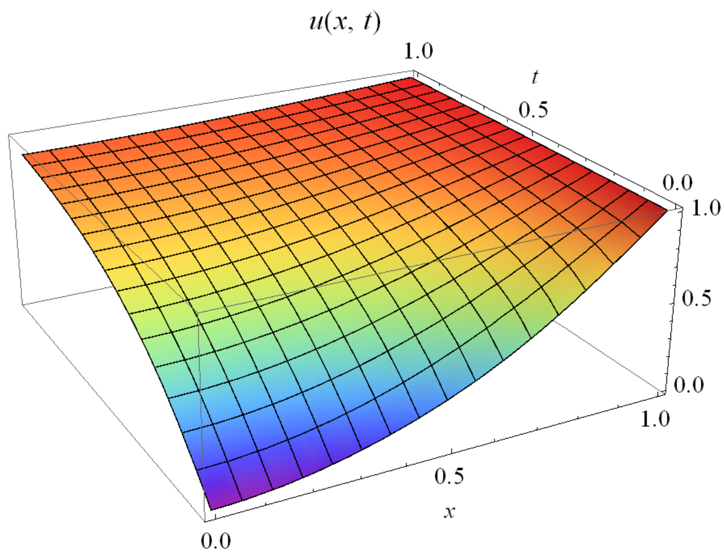

Figure 2: This is a plot of the two-dimensional solution surface $u(x, t)$ in three-dimensional space for $0 < x < 1$ and $0 < t < 1$.