

Exercise 2

For problem (1), complete the calculation of the series in case $j(t) = 0$ and $h(t) = e^t$.

Solution

With $j(t) = 0$ and $h(t) = e^t$, problem (1) on page 147 becomes

$$\begin{aligned}u_t &= ku_{xx} & 0 < x < l, & \quad t > 0 \\u(0, t) &= e^t & u(l, t) &= 0 \\u(x, 0) &= 0.\end{aligned}$$

The PDE is the heat equation for a homogeneous one-dimensional rod. The temperature at the left end rises exponentially in time, whereas the right end remains at 0° indefinitely. Initially the entire rod is at 0° .

Method 1 - Using Term-by-Term Differentiation

For this method to work, the boundary conditions must be made homogeneous. Make the change of variables $v(x, t) = u(x, t) - r(x, t)$, where $r(x, t)$ is some function that satisfies the boundary conditions.

$$\begin{aligned}r(0, t) &= e^t \\r(l, t) &= 0\end{aligned}$$

A suitable function is

$$r(x, t) = \frac{l-x}{l}e^t.$$

As a result, the initial and boundary conditions for v are

$$\begin{aligned}v(0, t) &= u(0, t) - r(0, t) = e^t - e^t = 0 \\v(l, t) &= u(l, t) - r(l, t) = 0 - 0 = 0 \\v(x, 0) &= u(x, 0) - r(x, 0) = 0 - \frac{l-x}{l} = \frac{x-l}{l}.\end{aligned}$$

Substitute $u(x, t) = v(x, t) + r(x, t)$ into the PDE now to find the one that v satisfies.

$$v_t + r_t = k(v_{xx} + r_{xx})$$

Distribute k and bring r_t to the right side.

$$v_t = kv_{xx} + kr_{xx} - r_t$$

Consequently, the PDE for v is

$$v_t = kv_{xx} + \frac{x-l}{l}e^t. \tag{1}$$

Since it is linear and inhomogeneous, we choose to apply the method of eigenfunction expansion to solve it. Consider the eigenvalue problem of the differential operator involving the spatial variable x

$$\frac{d^2}{dx^2}\phi = \lambda\phi \tag{2}$$

with the same boundary conditions as v .

$$\begin{aligned}\phi(0) &= 0 \\ \phi(l) &= 0\end{aligned}$$

Values of λ for which the boundary conditions are satisfied are known as the eigenvalues, and the nontrivial solutions associated with them are called the eigenfunctions. Equation (2) is known as the one-dimensional Helmholtz equation; the eigenfunctions for it are known to be orthogonal and form a complete set, which will prove useful later.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Suppose that λ is positive. Then equation (2) becomes

$$\frac{d^2\phi}{dx^2} = \mu^2\phi.$$

Its solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\phi(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Apply the boundary conditions to determine C_1 and C_2 .

$$\begin{aligned}\phi(0) &= C_1 = 0 \\ \phi(l) &= C_1 \cosh \mu l + C_2 \sinh \mu l = 0\end{aligned}$$

Since $C_1 = 0$, the second equation reduces to $C_2 \sinh \mu l = 0$. Hyperbolic sine is not oscillatory, so the only way this equation is satisfied is if $C_2 = 0$. The trivial solution is obtained, so there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Suppose that λ is zero. Then equation (2) becomes

$$\frac{d^2\phi}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\phi(x) = C_3 x + C_4$$

Apply the boundary conditions to determine C_3 and C_4 .

$$\begin{aligned}\phi(0) &= C_4 = 0 \\ \phi(l) &= C_3 l + C_4 = 0\end{aligned}$$

Since $C_4 = 0$, the second equation reduces to $C_3 = 0$. The trivial solution is obtained, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Suppose that λ is negative. Then equation (2) becomes

$$\frac{d^2\phi}{dx^2} = -\gamma^2\phi.$$

Its solution can be written in terms of sine and cosine.

$$\phi(x) = C_5 \cos \gamma x + C_6 \sin \gamma x$$

Apply the boundary conditions to determine C_5 and C_6 .

$$\begin{aligned}\phi(0) &= C_5 = 0 \\ \phi(l) &= C_5 \cos \gamma l + C_6 \sin \gamma l = 0\end{aligned}$$

Since $C_5 = 0$, the second equation reduces to $C_6 \sin \gamma l = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned}\sin \gamma l &= 0 \\ \gamma l &= n\pi, \quad n = 1, 2, \dots \\ \gamma_n &= \frac{n\pi}{l}, \quad n = 1, 2, \dots\end{aligned}$$

The eigenfunctions associated with these eigenvalues for λ are

$$\phi(x) = C_6 \sin \gamma x \quad \rightarrow \quad \phi_n(x) = \sin \frac{n\pi x}{l}, \quad n = 1, 2, \dots$$

and form a complete set, so the unknown function v can be expanded in terms of them.

$$v(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{l}$$

To determine $a_n(t)$, the generalized Fourier coefficients, substitute this formula for v into the PDE it satisfies, equation (1).

$$\frac{\partial}{\partial t} \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{l} = k \frac{\partial^2}{\partial x^2} \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{l} + \frac{x-l}{l} e^t$$

Because v satisfies homogeneous boundary conditions and v , $\partial v/\partial x$, and $\partial v/\partial t$ are continuous (reasonable assumptions for the temperature profile in a homogeneous solid rod), the two series can in fact be differentiated term by term.

$$\sum_{n=1}^{\infty} \frac{da_n}{dt} \sin \frac{n\pi x}{l} = k \sum_{n=1}^{\infty} a_n(t) \frac{d^2}{dx^2} \sin \frac{n\pi x}{l} + \frac{x-l}{l} e^t$$

The operator applied to the eigenfunction is equal to the eigenvalue times the eigenfunction.

$$\sum_{n=1}^{\infty} \frac{da_n}{dt} \sin \frac{n\pi x}{l} = k \sum_{n=1}^{\infty} a_n(t) \lambda_n \sin \frac{n\pi x}{l} + \frac{x-l}{l} e^t$$

Bring both series to the left side and combine them.

$$\sum_{n=1}^{\infty} \left[\frac{da_n}{dt} - ka_n(t)\lambda_n \right] \sin \frac{n\pi x}{l} = \frac{x-l}{l} e^t$$

The left side is essentially a Fourier sine series expansion of the function on the right side. To solve for the term enclosed in square brackets, multiply both sides by $\sin m\pi x/l$, where m is an integer,

$$\sum_{n=1}^{\infty} \left[\frac{da_n}{dt} - ka_n(t)\lambda_n \right] \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} = \frac{x-l}{l} e^t \sin \frac{m\pi x}{l}$$

and then integrate both sides with respect to x from 0 to l .

$$\int_0^l \sum_{n=1}^{\infty} \left[\frac{da_n}{dt} - ka_n(t)\lambda_n \right] \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \int_0^l \frac{x-l}{l} e^t \sin \frac{m\pi x}{l} dx$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty} \left[\frac{da_n}{dt} - ka_n(t)\lambda_n \right] \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \frac{e^t}{l} \int_0^l (x-l) \sin \frac{m\pi x}{l} dx$$

Because the sine functions are orthogonal, the integral on the left side is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$\left[\frac{da_n}{dt} - ka_n(t)\lambda_n \right] \int_0^l \sin^2 \frac{n\pi x}{l} dx = \frac{e^t}{l} \int_0^l (x-l) \sin \frac{n\pi x}{l} dx$$

Evaluate the integrals.

$$\left[\frac{da_n}{dt} - ka_n(t)\lambda_n \right] \cdot \frac{l}{2} = \frac{e^t}{l} \left(-\frac{l^2}{n\pi} \right)$$

Multiply both sides by $2/l$ and simplify both sides.

$$\frac{da_n}{dt} - k\lambda_n a_n = -\frac{2e^t}{n\pi} \quad (3)$$

With the help of the method of eigenfunction expansion, the PDE for v has been reduced to a first-order inhomogeneous ODE, which can be solved with an integrating factor I .

$$I = \exp \left[\int^t (-k\lambda_n) ds \right] = e^{-k\lambda_n t}$$

Multiply both sides of the previous equation by I .

$$\frac{da_n}{dt} e^{-k\lambda_n t} - k\lambda_n a_n e^{-k\lambda_n t} = -\frac{2e^t}{n\pi} e^{-k\lambda_n t}$$

The left side can be written as $d/dt(Ia_n)$ by the product rule. Combine the exponentials on the right side.

$$\frac{d}{dt} (e^{-k\lambda_n t} a_n) = -\frac{2}{n\pi} e^{-(k\lambda_n - 1)t}$$

Integrate both sides with respect to t .

$$e^{-k\lambda_n t} a_n = \int^t \left(-\frac{2}{n\pi} \right) e^{-(k\lambda_n - 1)s} ds + C_7$$

Evaluate the remaining integral.

$$e^{-k\lambda_n t} a_n = \frac{2}{n\pi(k\lambda_n - 1)} e^{-(k\lambda_n - 1)t} + C_7$$

The general solution for a_n is thus

$$a_n(t) = \frac{2}{n\pi(k\lambda_n - 1)} e^t + C_7 e^{k\lambda_n t}. \quad (4)$$

In order to determine C_7 , use the initial condition for v in combination with the eigenfunction expansion.

$$v(x, 0) = \sum_{n=1}^{\infty} a_n(0) \sin \frac{n\pi x}{l} = \frac{x-l}{l}$$

To solve for $a_n(0)$, multiply both sides by $\sin(m\pi x/l)$

$$\sum_{n=1}^{\infty} a_n(0) \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} = \frac{x-l}{l} \sin \frac{m\pi x}{l}$$

and then integrate both sides with respect to x from 0 to l .

$$\int_0^l \sum_{n=1}^{\infty} a_n(0) \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \int_0^l \frac{x-l}{l} \sin \frac{m\pi x}{l} dx$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty} a_n(0) \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \frac{1}{l} \int_0^l (x-l) \sin \frac{m\pi x}{l} dx$$

The orthogonality of the sine functions results in the disappearance of every term in the infinite series except the $n = m$ term.

$$a_n(0) \int_0^l \sin^2 \frac{n\pi x}{l} dx = \frac{1}{l} \int_0^l (x-l) \sin \frac{n\pi x}{l} dx$$

Evaluate the integrals.

$$a_n(0) \cdot \frac{l}{2} = \frac{1}{l} \left(-\frac{l^2}{n\pi} \right)$$

Solve for $a_n(0)$.

$$a_n(0) = -\frac{2}{n\pi}$$

Now set $t = 0$ in equation (4) and use this initial condition to find C_7 .

$$a_n(0) = \frac{2}{n\pi(k\lambda_n - 1)} + C_7 = -\frac{2}{n\pi}$$

Solving for it gives

$$C_7 = -\frac{2k\lambda_n}{n\pi(k\lambda_n - 1)}.$$

So then

$$\begin{aligned} a_n(t) &= \frac{2}{n\pi(k\lambda_n - 1)}e^t - \frac{2k\lambda_n}{n\pi(k\lambda_n - 1)}e^{k\lambda_n t} \\ &= \frac{2}{n\pi(k\lambda_n - 1)}\left(e^t - k\lambda_n e^{k\lambda_n t}\right). \end{aligned}$$

Substitute $\lambda_n = -\gamma_n^2 = -(n\pi/l)^2$ here.

$$\begin{aligned} &= \frac{2}{n\pi\left(-k\frac{n^2\pi^2}{l^2} - 1\right)}\left[e^t + k\frac{n^2\pi^2}{l^2}\exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\right] \\ &= \frac{-2l^2}{n\pi(kn^2\pi^2 + l^2)}\left[e^t + k\frac{n^2\pi^2}{l^2}\exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\right] \end{aligned}$$

Consequently,

$$v(x, t) = \sum_{n=1}^{\infty} \frac{-2l^2}{n\pi(kn^2\pi^2 + l^2)}\left[e^t + k\frac{n^2\pi^2}{l^2}\exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\right] \sin \frac{n\pi x}{l}$$

and, since $u(x, t) = v(x, t) + r(x, t)$,

$$u(x, t) = \frac{l-x}{l}e^t + \sum_{n=1}^{\infty} \frac{-2l^2}{n\pi(kn^2\pi^2 + l^2)}\left[e^t + k\frac{n^2\pi^2}{l^2}\exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\right] \sin \frac{n\pi x}{l}.$$

Therefore,

$$u(x, t) = \frac{l-x}{l}e^t - \frac{2l^2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n(kn^2\pi^2 + l^2)}\left[e^t + k\frac{n^2\pi^2}{l^2}\exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\right] \sin \frac{n\pi x}{l}, \quad 0 \leq x \leq l.$$

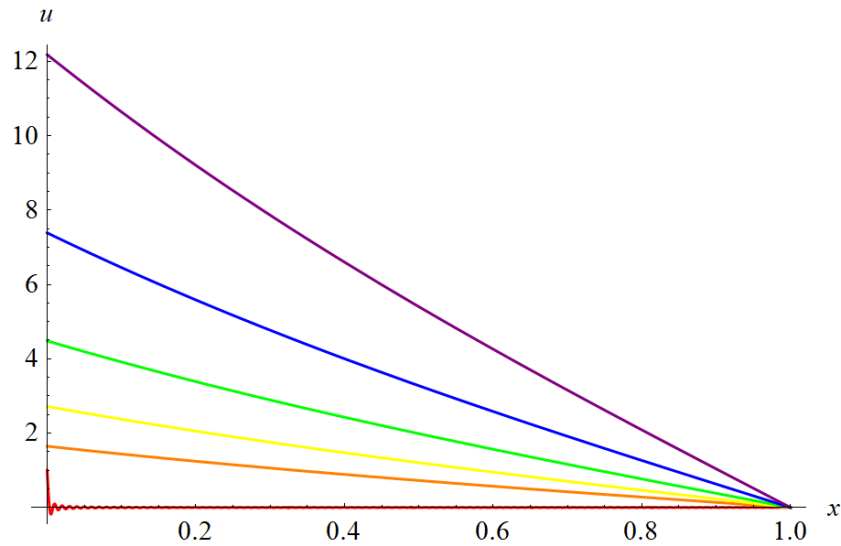


Figure 1: This figure shows $u(x, t)$ at various times for $k = 1$ and $l = 1$. The curves in red, orange, yellow, green, blue, and purple correspond to $t = 0$, $t = 0.5$, $t = 1$, $t = 1.5$, $t = 2$, and $t = 2.5$, respectively. They are approximate, as only the first 200 terms in the infinite series have been used.

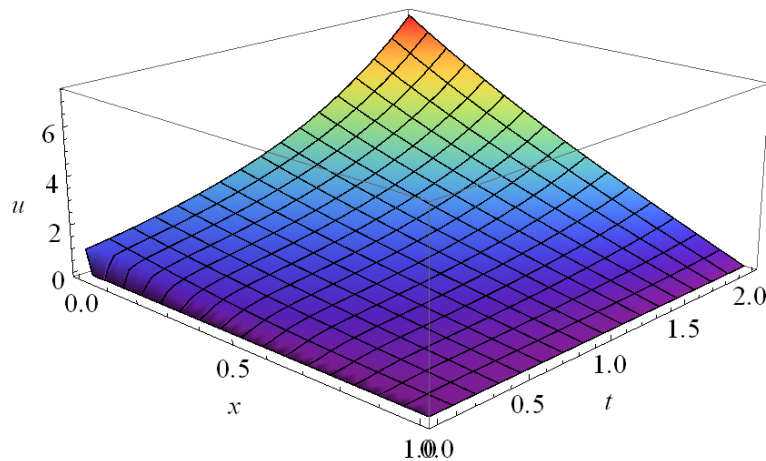


Figure 2: This is a plot of the two-dimensional solution surface $u(x, t)$ in three-dimensional xyt -space for $k = 1$ and $l = 1$. It is approximate, as only the first 200 terms in the infinite series have been used.

Method 2 - Mr. Strauss's Way

The method of eigenfunction expansion will be applied directly to solve for u as Mr. Strauss does in the textbook, that is, without making the boundary conditions homogeneous. The same eigenvalue problem is considered here as before.

$$\frac{d^2}{dx^2}\phi = \lambda\phi, \quad \begin{aligned} \phi(0) &= 0 \\ \phi(l) &= 0 \end{aligned}$$

It was shown that the eigenvalues are $\lambda_n = -(n\pi/l)^2$ and that the eigenfunctions associated with them are

$$\phi_n(x) = \sin \frac{n\pi x}{l}, \quad n = 1, 2, \dots$$

The eigenfunctions of the Helmholtz equation are known to form a complete set, so all of the functions in the PDE can be expanded in terms of them.

$$\begin{aligned} u(x, t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x) &\rightarrow u\phi_m = \sum_{n=1}^{\infty} b_n\phi_n\phi_m &\rightarrow \int_0^l u\phi_n dx = b_n \int_0^l \phi_n^2 dx = b_n \cdot \frac{l}{2} \\ \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} c_n(t)\phi_n(x) &\rightarrow \frac{\partial u}{\partial t}\phi_m = \sum_{n=1}^{\infty} c_n\phi_n\phi_m &\rightarrow \int_0^l \frac{\partial u}{\partial t}\phi_n dx = c_n \int_0^l \phi_n^2(x) dx = c_n \cdot \frac{l}{2} \\ \frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} d_n(t)\phi_n(x) &\rightarrow \frac{\partial^2 u}{\partial x^2}\phi_m = \sum_{n=1}^{\infty} d_n\phi_n\phi_m &\rightarrow \int_0^l \frac{\partial^2 u}{\partial x^2}\phi_n dx = d_n \int_0^l \phi_n^2(x) dx = d_n \cdot \frac{l}{2} \end{aligned}$$

It should be emphasized that these are generalized Fourier series expansions for u , $\partial u/\partial t$, and $\partial^2 u/\partial x^2$, not product solutions that come about from using the method of separation of variables. Solve the latter equations for the generalized Fourier coefficients.

$$\begin{aligned} b_n(t) &= \frac{2}{l} \int_0^l u\phi_n dx \\ c_n(t) &= \frac{2}{l} \int_0^l \frac{\partial u}{\partial t}\phi_n dx = \frac{d}{dt} \left(\frac{2}{l} \int_0^l u\phi_n dx \right) = \frac{db_n}{dt} \\ d_n(t) &= \frac{2}{l} \int_0^l \frac{\partial^2 u}{\partial x^2}\phi_n dx = \frac{2}{l} \left(\underbrace{\frac{\partial u}{\partial x}\phi_n \Big|_0^l}_{=0} - \int_0^l \frac{\partial u}{\partial x} \frac{d\phi_n}{dx} dx \right) = -\frac{2n\pi}{l^2} \int_0^l \frac{\partial u}{\partial x} \cos \frac{n\pi x}{l} dx \end{aligned}$$

Apply integration by parts once more in order to write d_n in terms of b_n and the given boundary conditions for u .

$$\begin{aligned} d_n(t) &= -\frac{2n\pi}{l^2} \left[u(x, t) \cos \frac{n\pi x}{l} \Big|_0^l - \int_0^l u \left(-\frac{n\pi}{l} \right) \sin \frac{n\pi x}{l} dx \right] \\ &= -\frac{2n\pi}{l^2} \left[u(l, t) \cos n\pi - u(0, t) + \frac{n\pi}{l} \int_0^l u \sin \frac{n\pi x}{l} dx \right] \\ &= -\frac{2n\pi}{l^2} \left[-e^t + \frac{n\pi}{2} \left(\frac{2}{l} \int_0^l u \sin \frac{n\pi x}{l} dx \right) \right] \\ &= -\frac{2n\pi}{l^2} \left(-e^t + \frac{n\pi}{2} b_n \right) \\ &= \frac{n\pi}{l^2} (2e^t - n\pi b_n) \end{aligned}$$

Now that the coefficients are known, substitute the eigenfunction expansions into the PDE.

$$\begin{aligned}
 u_t &= ku_{xx} \\
 \sum_{n=1}^{\infty} c_n(t)\phi_n(x) &= k \sum_{n=1}^{\infty} d_n(t)\phi_n(x) \\
 \sum_{n=1}^{\infty} \frac{db_n}{dt}\phi_n(x) &= \sum_{n=1}^{\infty} \frac{kn\pi}{l^2}(2e^t - n\pi b_n)\phi_n(x)
 \end{aligned}$$

The summands must be equal as a result.

$$\frac{db_n}{dt}\phi_n(x) = \frac{kn\pi}{l^2}(2e^t - n\pi b_n)\phi_n(x)$$

Divide both sides by $\phi_n(x)$.

$$\frac{db_n}{dt} = \frac{kn\pi}{l^2}(2e^t - n\pi b_n)$$

Expand the right side and bring the term with b_n to the left side.

$$\frac{db_n}{dt} + \frac{kn^2\pi^2}{l^2}b_n = \frac{2kn\pi}{l^2}e^t$$

With the help of the method of eigenfunction expansion, the PDE has been reduced to a first-order inhomogeneous ODE that can be solved with an integrating factor I_2 .

$$I_2 = \exp\left(\int^t \frac{kn^2\pi^2}{l^2} ds\right) = \exp\left(\frac{kn^2\pi^2}{l^2}t\right)$$

Multiply both sides by I_2 .

$$\frac{db_n}{dt} \exp\left(\frac{kn^2\pi^2}{l^2}t\right) + \frac{kn^2\pi^2}{l^2}b_n \exp\left(\frac{kn^2\pi^2}{l^2}t\right) = \frac{2kn\pi}{l^2}e^t \exp\left(\frac{kn^2\pi^2}{l^2}t\right)$$

The left side can be written as $d/dt(I_2b_n)$ by the product rule.

$$\frac{d}{dt} \left[\exp\left(\frac{kn^2\pi^2}{l^2}t\right) b_n \right] = \frac{2kn\pi}{l^2} \exp\left[\left(\frac{kn^2\pi^2}{l^2} + 1\right)t\right]$$

Integrate both sides with respect to t .

$$\exp\left(\frac{kn^2\pi^2}{l^2}t\right) b_n = \int^t \frac{2kn\pi}{l^2} \exp\left[\left(\frac{kn^2\pi^2}{l^2} + 1\right)s\right] ds + C_8$$

Evaluate the integral.

$$\exp\left(\frac{kn^2\pi^2}{l^2}t\right) b_n = \frac{2kn\pi}{l^2} \frac{1}{\frac{kn^2\pi^2}{l^2} + 1} \exp\left[\left(\frac{kn^2\pi^2}{l^2} + 1\right)t\right] + C_8$$

Divide both sides by I_2 to solve for b_n .

$$b_n(t) = \frac{2kn\pi}{kn^2\pi^2 + l^2}e^t + C_8 \exp\left(-\frac{kn^2\pi^2}{l^2}t\right)$$

In order to determine C_8 , use the initial condition for u in combination with the eigenfunction expansion.

$$u(x, 0) = \sum_{n=1}^{\infty} b_n(0) \phi_n(x) = 0 \quad \rightarrow \quad b_n(0) = 0$$

Set $t = 0$ in the previous equation and use the initial condition.

$$b_n(0) = \frac{2kn\pi}{kn^2\pi^2 + l^2} + C_8 = 0 \quad \rightarrow \quad C_8 = -\frac{2kn\pi}{kn^2\pi^2 + l^2}$$

So then

$$\begin{aligned} b_n(t) &= \frac{2kn\pi}{kn^2\pi^2 + l^2} e^t - \frac{2kn\pi}{kn^2\pi^2 + l^2} \exp\left(-\frac{kn^2\pi^2}{l^2}t\right) \\ &= \frac{2kn\pi}{kn^2\pi^2 + l^2} \left[e^t - \exp\left(-\frac{kn^2\pi^2}{l^2}t\right) \right]. \end{aligned}$$

Therefore,

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2kn\pi}{kn^2\pi^2 + l^2} \left[e^t - \exp\left(-\frac{kn^2\pi^2}{l^2}t\right) \right] \sin \frac{n\pi x}{l}, \quad 0 < x \leq l.$$

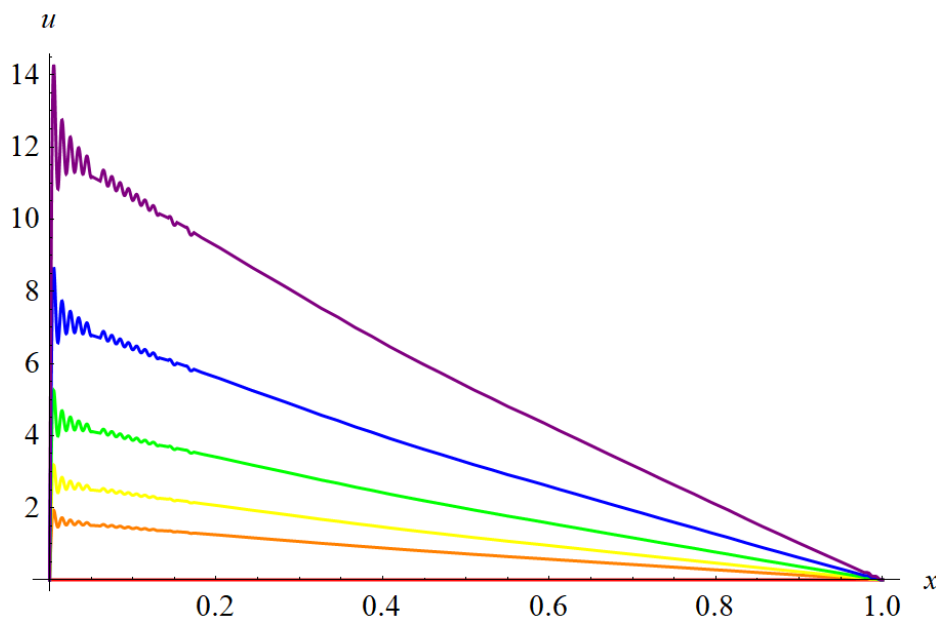


Figure 3: This figure shows $u(x, t)$ at various times for $k = 1$ and $l = 1$. The curves in red, orange, yellow, green, blue, and purple correspond to $t = 0$, $t = 0.5$, $t = 1$, $t = 1.5$, $t = 2$, and $t = 2.5$, respectively. They are approximate, as only the first 200 terms in the infinite series have been used.

Comparing Figure 3 with Figure 1, we conclude that applying the method of eigenfunction expansion directly as Mr. Strauss does in the textbook results in a series solution that not only converges very slowly but also does not satisfy the inhomogeneous boundary condition at $x = 0$. These issues stem from the fact that $\phi(0)$ and $u(0, t)$ are not equal; thus, they can be avoided by making the boundary conditions homogeneous first. This will be done next in Method 3.

Method 3 - Without Using Term-by-Term Differentiation

As in Method 1 the boundary conditions will be made homogeneous by making the substitution $v(x, t) = u(x, t) - r(x, t)$, where $r(x, t)$ is some function that satisfies the boundary conditions. If we choose

$$r(x, t) = \frac{l-x}{l}e^t,$$

then the initial boundary value problem that v was shown to satisfy is

$$\begin{aligned}v_t &= kv_{xx} + \frac{x-l}{l}e^t, & 0 < x < l, & \quad t > 0 \\v(0, t) &= 0 & v(l, t) &= 0 \\v(x, 0) &= \frac{x-l}{l}.\end{aligned}$$

The same eigenvalue problem is considered here.

$$\begin{aligned}\frac{d^2}{dx^2}\phi &= \lambda\phi, & \phi(0) &= 0 \\& & \phi(l) &= 0\end{aligned}$$

It was shown that the eigenvalues are $\lambda_n = -(n\pi/l)^2$ and that the eigenfunctions associated with them are

$$\phi_n(x) = \sin \frac{n\pi x}{l}, \quad n = 1, 2, \dots$$

The eigenfunctions of the Helmholtz equation are known to form a complete set, so all of the functions in the PDE can be expanded in terms of them.

$$\begin{aligned}v(x, t) &= \sum_{n=1}^{\infty} A_n(t)\phi_n(x) & \rightarrow & \quad v\phi_m = \sum_{n=1}^{\infty} A_n\phi_n\phi_m & \rightarrow & \quad \int_0^l v\phi_n dx = A_n \int_0^l \phi_n^2 dx = A_n \cdot \frac{l}{2} \\ \frac{\partial v}{\partial t} &= \sum_{n=1}^{\infty} B_n(t)\phi_n(x) & \rightarrow & \quad \frac{\partial v}{\partial t}\phi_m = \sum_{n=1}^{\infty} B_n\phi_n\phi_m & \rightarrow & \quad \int_0^l \frac{\partial v}{\partial t}\phi_n dx = B_n \int_0^l \phi_n^2 dx = B_n \cdot \frac{l}{2} \\ \frac{x-l}{l}e^t &= \sum_{n=1}^{\infty} D_n(t)\phi_n(x) & \rightarrow & \quad \frac{x-l}{l}e^t\phi_m = \sum_{n=1}^{\infty} D_n\phi_n\phi_m & \rightarrow & \quad \frac{e^t}{l} \int_0^l (x-l)\phi_n dx = D_n \int_0^l \phi_n^2 dx = D_n \cdot \frac{l}{2} \\ \frac{\partial^2 v}{\partial x^2} &= \sum_{n=1}^{\infty} E_n(t)\phi_n(x) & \rightarrow & \quad \frac{\partial^2 v}{\partial x^2}\phi_m = \sum_{n=1}^{\infty} E_n\phi_n\phi_m & \rightarrow & \quad \int_0^l \frac{\partial^2 v}{\partial x^2}\phi_n dx = E_n \int_0^l \phi_n^2 dx = E_n \cdot \frac{l}{2}\end{aligned}$$

It should be emphasized that these are generalized Fourier series expansions for the functions, not product solutions that come about from using the method of separation of variables. Solve the latter equations for the generalized Fourier coefficients.

$$\begin{aligned}A_n(t) &= \frac{2}{l} \int_0^l v\phi_n dx \\ B_n(t) &= \frac{2}{l} \int_0^l \frac{\partial v}{\partial t}\phi_n dx = \frac{d}{dt} \left(\frac{2}{l} \int_0^l v\phi_n dx \right) = \frac{dA_n}{dt} \\ D_n(t) &= \frac{2e^t}{l^2} \int_0^l (x-l)\phi_n dx = \frac{2e^t}{l^2} \left(-\frac{l^2}{n\pi} \right) = -\frac{2e^t}{n\pi} \\ E_n(t) &= \frac{2}{l} \int_0^l \frac{\partial^2 v}{\partial x^2}\phi_n dx = \frac{2}{l} \left(\underbrace{\frac{\partial v}{\partial x}\phi_n \Big|_0^l}_{=0} - \int_0^l \frac{\partial v}{\partial x} \frac{d\phi_n}{dx} dx \right) = -\frac{2n\pi}{l^2} \int_0^l \frac{\partial v}{\partial x} \cos \frac{n\pi x}{l} dx\end{aligned}$$

Apply integration by parts once more in order to write E_n in terms of A_n and the boundary conditions for v .

$$\begin{aligned}
 E_n(t) &= -\frac{2n\pi}{l^2} \left[v(x, t) \cos \frac{n\pi x}{l} \Big|_0^l - \int_0^l v \left(-\frac{n\pi}{l} \right) \sin \frac{n\pi x}{l} dx \right] \\
 &= -\frac{2n\pi}{l^2} \left[v(l, t) \cos n\pi - v(0, t) + \frac{n\pi}{l} \int_0^l v \sin \frac{n\pi x}{l} dx \right] \\
 &= -\frac{2n\pi}{l^2} \left[\frac{n\pi}{2} \left(\frac{2}{l} \int_0^l v \sin \frac{n\pi x}{l} dx \right) \right] \\
 &= -\frac{2n\pi}{l^2} \left(\frac{n\pi}{2} A_n \right) \\
 &= -\frac{n^2\pi^2}{l^2} A_n
 \end{aligned}$$

Now that the coefficients are known, substitute the eigenfunction expansions into the PDE.

$$\begin{aligned}
 v_t &= kv_{xx} + \frac{x-l}{l} e^t \\
 \sum_{n=1}^{\infty} B_n(t) \phi_n(x) &= k \sum_{n=1}^{\infty} E_n(t) \phi_n(x) + \sum_{n=1}^{\infty} D_n(t) \phi_n(x) \\
 \sum_{n=1}^{\infty} \frac{dA_n}{dt} \phi_n(x) &= k \sum_{n=1}^{\infty} \left(-\frac{n^2\pi^2}{l^2} A_n \right) \phi_n(x) + \sum_{n=1}^{\infty} \left(-\frac{2e^t}{n\pi} \right) \phi_n(x) \\
 \sum_{n=1}^{\infty} \frac{dA_n}{dt} \phi_n(x) &= \sum_{n=1}^{\infty} \left(-k \frac{n^2\pi^2}{l^2} A_n - \frac{2e^t}{n\pi} \right) \phi_n(x)
 \end{aligned}$$

The summands must be equal as a result.

$$\frac{dA_n}{dt} \phi_n(x) = \left(-k \frac{n^2\pi^2}{l^2} A_n - \frac{2e^t}{n\pi} \right) \phi_n(x)$$

Divide both sides by $\phi_n(x)$ and bring the term with A_n to the left side.

$$\frac{dA_n}{dt} + k \frac{n^2\pi^2}{l^2} A_n = -\frac{2e^t}{n\pi}$$

This is the same ODE that a_n satisfies in equation (3) since $\lambda_n = -(n\pi/l)^2$, so A_n has the same general solution. Also, the initial condition for v is the same, so the particular solution is the same.

$$A_n(t) = \frac{-2l^2}{n\pi(kn^2\pi^2 + l^2)} \left[e^t + k \frac{n^2\pi^2}{l^2} \exp \left(-k \frac{n^2\pi^2}{l^2} t \right) \right]$$

Therefore,

$$v(x, t) = \sum_{n=1}^{\infty} \frac{-2l^2}{n\pi(kn^2\pi^2 + l^2)} \left[e^t + k \frac{n^2\pi^2}{l^2} \exp \left(-k \frac{n^2\pi^2}{l^2} t \right) \right] \sin \frac{n\pi x}{l}$$

and, because $u(x, t) = v(x, t) + r(x, t)$,

$$u(x, t) = \frac{l-x}{l} e^t - \frac{2l^2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n(kn^2\pi^2 + l^2)} \left[e^t + k \frac{n^2\pi^2}{l^2} \exp \left(-k \frac{n^2\pi^2}{l^2} t \right) \right] \sin \frac{n\pi x}{l}, \quad 0 \leq x \leq l.$$