Exercise 4

Solve $u_{tt} = c^2 u_{xx} + k$ for $0 < x < l$, with the boundary conditions $u(0,t) = 0$, $u_x(l,t) = 0$ and the initial conditions $u(x,0) = 0$, $u_t(x,0) = V$. Here k and V are constants.

Solution

Since the PDE is linear and inhomogeneous, we choose to apply the method of eigenfunction expansion to solve it. Consider the eigenvalue problem of the differential operator involving the spatial variable x

$$\frac{d^2}{dx^2}\phi = \lambda\phi \tag{1}$$

with the same boundary conditions as u.

$$\begin{aligned} \phi(0) &= 0 \\ \phi'(l) &= 0 \end{aligned}$$

Values of λ for which the boundary conditions are satisfied are known as the eigenvalues, and the nontrivial solutions associated with them are called the eigenfunctions. Equation (1) is known as the one-dimensional Helmholtz equation; the eigenfunctions for it are known to be orthogonal and form a complete set, which will prove useful later.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Suppose that λ is positive. Then equation (1) becomes

$$\frac{d^2\phi}{dx^2} = \mu^2\phi.$$

Its solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\begin{aligned} \phi(x) &= C_1 \cosh \mu x + C_2 \sinh \mu x \\ \phi'(x) &= \mu(C_1 \sinh \mu x + C_2 \cosh \mu x) \end{aligned}$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} \phi(0) &= C_1 = 0 \\ \phi'(l) &= \mu(C_1 \sinh \mu l + C_2 \cosh \mu l) = 0 \end{aligned}$$

Since $C_1 = 0$, the second equation reduces to $\mu C_2 \cosh \mu l = 0$. Hyperbolic cosine is not oscillatory, so the only way this equation is satisfied is if $C_2 = 0$. The trivial solution is obtained, so there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Suppose that λ is zero. Then equation (1) becomes

$$\frac{d^2\phi}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\phi(x) = C_3 x + C_4$$

Apply the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} \phi(0) &= C_4 = 0 \\ \phi'(l) &= C_3 = 0 \end{aligned}$$

The trivial solution is obtained, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Suppose that λ is negative. Then equation (1) becomes

$$\frac{d^2\phi}{dx^2} = -\gamma^2\phi.$$

Its solution can be written in terms of sine and cosine.

$$\begin{aligned} \phi(x) &= C_5 \cos\gamma x + C_6 \sin\gamma x \\ \phi'(x) &= \gamma(-C_5 \sin\gamma x + C_6 \cos\gamma x) \end{aligned}$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} \phi(0) &= C_5 = 0 \\ \phi'(l) &= \gamma(-C_5 \sin\gamma l + C_6 \cos\gamma l) = 0 \end{aligned}$$

Since $C_5 = 0$, the second equation reduces to $\gamma C_6 \cos\gamma l = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned} \cos\gamma l &= 0 \\ \gamma l &= \frac{\pi}{2}(2n-1), \quad n = 1, 2, \ldots \\ \gamma_n &= \frac{\pi}{2l}(2n-1), \quad n = 1, 2, \ldots. \end{aligned}$$

The eigenfunctions associated with these eigenvalues for λ are

$$\phi(x) = C_6 \sin\gamma x \quad \rightarrow \quad \phi_n(x) = \sin\left[\frac{\pi}{2l}(2n-1)x\right], \quad n = 1, 2, \ldots.$$

Method 1 - Using Term-by-Term Differentiation

The eigenfunctions of the Helmholtz equation form a complete set, so the unknown function u can be expanded in terms of them.

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left[\frac{\pi}{2l}(2n-1)x\right]$$

To determine the generalized Fourier coefficients $a_n(t)$, substitute this expansion into the PDE.

$$u_{tt} = c^2 u_{xx} + k$$

$$\frac{\partial^2}{\partial t^2} \sum_{n=1}^{\infty} a_n(t) \sin\left[\frac{\pi}{2l}(2n-1)x\right] = c^2 \frac{\partial^2}{\partial x^2} \sum_{n=1}^{\infty} a_n(t) \sin\left[\frac{\pi}{2l}(2n-1)x\right] + k$$

Because u satisfies homogeneous boundary conditions and u, $\partial u/\partial x$, and $\partial u/\partial t$ are continuous (reasonable assumptions for the displacement of a homogeneous elastic string), the two series can in fact be differentiated term by term.

$$\sum_{n=1}^{\infty} \frac{d^2 a_n}{dt^2} \sin\left[\frac{\pi}{2l}(2n-1)x\right] = c^2 \sum_{n=1}^{\infty} a_n(t) \frac{d^2}{dx^2} \sin\left[\frac{\pi}{2l}(2n-1)x\right] + k$$

The operator applied to the eigenfunction is equal to the eigenvalue times the eigenfunction.

$$\sum_{n=1}^{\infty} \frac{d^2 a_n}{dt^2} \sin\left[\frac{\pi}{2l}(2n-1)x\right] = c^2 \sum_{n=1}^{\infty} a_n(t)\lambda_n \sin\left[\frac{\pi}{2l}(2n-1)x\right] + k$$

Bring both series to the left side and combine them.

$$\sum_{n=1}^{\infty} \left[\frac{d^2 a_n}{dt^2} - c^2 \lambda_n a_n(t)\right] \sin\left[\frac{\pi}{2l}(2n-1)x\right] = k$$

To solve for the coefficient of sine, multiply both sides by $\sin \gamma_m$, where m is an integer,

$$\sum_{n=1}^{\infty} \left[\frac{d^2 a_n}{dt^2} - c^2 \lambda_n a_n(t)\right] \sin\left[\frac{\pi}{2l}(2n-1)x\right] \sin\left[\frac{\pi}{2l}(2m-1)x\right] = k \sin\left[\frac{\pi}{2l}(2m-1)x\right]$$

and then integrate both sides with respect to x from 0 to l.

$$\int_0^l \sum_{n=1}^{\infty} \left[\frac{d^2 a_n}{dt^2} - c^2 \lambda_n a_n(t)\right] \sin\left[\frac{\pi}{2l}(2n-1)x\right] \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx = \int_0^l k \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty} \left[\frac{d^2 a_n}{dt^2} - c^2 \lambda_n a_n(t)\right] \int_0^l \sin\left[\frac{\pi}{2l}(2n-1)x\right] \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx = k \int_0^l \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx$$

Because the eigenfunctions are orthogonal, the integral on the left side is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$\left[\frac{d^2 a_n}{dt^2} - c^2 \lambda_n a_n(t)\right] \int_0^l \sin^2\left[\frac{\pi}{2l}(2n-1)x\right] dx = k \int_0^l \sin\left[\frac{\pi}{2l}(2n-1)x\right] dx$$

Evaluate the integrals.

$$\left[\frac{d^2 a_n}{dt^2} - c^2 \lambda_n a_n(t)\right] \cdot \frac{l}{2} = k \cdot \frac{2l}{\pi(2n-1)}$$

Multiply both sides by $2/l$ and replace λ_n with $-\gamma_n^2$.

$$\frac{d^2 a_n}{dt^2} + c^2 \gamma_n^2 a_n = \frac{4k}{\pi(2n-1)}$$

As a consequence of using the method of eigenfunction expansion, the PDE has reduced to a second-order linear inhomogeneous ODE. Since the ODE is linear, the general solution is the sum of a complementary solution and a particular solution.

$$a_n = a_c + a_p$$

The complementary solution satisfies the associated homogeneous equation.

$$\frac{d^2a_c}{dt^2} + c^2\gamma_n^2 a_c = 0$$

The coefficients are independent of time, so the solutions are of the form $a_c = e^{rt}$. Take two derivatives of it

$$a_c = e^{rt} \quad \to \quad \frac{da_c}{dt} = re^{rt} \quad \to \quad \frac{d^2a_c}{dt^2} = r^2e^{rt}$$

and substitute these formulas into the equation to find r.

$$r^2e^{rt} + c^2\gamma_n^2 e^{rt} = 0$$

$$r^2 + c^2\gamma_n^2 = 0$$

$$r = \pm ic\gamma_n$$

So then

$$a_c(t) = C_7e^{ic\gamma_n t} + C_8e^{-ic\gamma_n t}.$$

With a different choice of constants, the solution can be written in terms of sine and cosine.

$$a_c(t) = C_9 \cos c\gamma_n t + C_{10} \sin c\gamma_n t$$

Because the inhomogeneous term is constant in time, the particular solution is also a constant. d^2a_p/dt^2 vanishes as a result.

$$\underbrace{\frac{d^2a_p}{dt^2}}_{=\,0} + c^2\gamma_n^2 a_p = \frac{4k}{\pi(2n-1)} \quad \to \quad a_p(t) = \frac{1}{c^2\gamma_n^2}\frac{4k}{\pi(2n-1)}$$

Thus, the general solution for a_n is

$$a_n(t) = C_9 \cos c\gamma_n t + C_{10} \sin c\gamma_n t + \frac{1}{c^2\gamma_n^2}\frac{4k}{\pi(2n-1)}.$$

Use the initial conditions for u in combination with its eigenfunction expansion to obtain the initial conditions for a_n.

$$u(x,0) = \sum_{n=1}^{\infty} a_n(0) \sin\left[\frac{\pi}{2l}(2n-1)x\right] = 0$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{da_n}{dt}(0) \sin\left[\frac{\pi}{2l}(2n-1)x\right] = V$$

We have $a_n(0) = 0$ from the first equation. To solve the second equation for $da_n/dt(0)$, multiply both sides by $\sin \gamma_m$

$$\sum_{n=1}^{\infty} \frac{da_n}{dt}(0) \sin\left[\frac{\pi}{2l}(2n-1)x\right] \sin\left[\frac{\pi}{2l}(2m-1)x\right] = V \sin\left[\frac{\pi}{2l}(2m-1)x\right]$$

and then integrate both sides with respect to x from 0 to l.

$$\int_0^l \sum_{n=1}^{\infty} \frac{da_n}{dt}(0) \sin\left[\frac{\pi}{2l}(2n-1)x\right] \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx = \int_0^l V \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty} \frac{da_n}{dt}(0) \int_0^l \sin\left[\frac{\pi}{2l}(2n-1)x\right] \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx = V \int_0^l \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx$$

Because the eigenfunctions are orthogonal, the integral on the left side is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$\frac{da_n}{dt}(0) \int_0^l \sin^2\left[\frac{\pi}{2l}(2n-1)x\right] dx = V \int_0^l \sin\left[\frac{\pi}{2l}(2n-1)x\right] dx$$

Evaluate the integrals.

$$\frac{da_n}{dt}(0) \cdot \frac{l}{2} = V \cdot \frac{2l}{\pi(2n-1)}$$

So the two initial conditions for a_n are

$$\begin{aligned} a_n(0) &= 0 \\ \frac{da_n}{dt}(0) &= \frac{4V}{\pi(2n-1)}. \end{aligned}$$

Applying them, we get

$$\begin{aligned} a_n(0) = C_9 + \frac{1}{c^2\gamma_n^2}\frac{4k}{\pi(2n-1)} = 0 \qquad &\to \qquad C_9 = -\frac{1}{c^2\gamma_n^2}\frac{4k}{\pi(2n-1)} \\ \frac{da_n}{dt}(0) = c\gamma_n(C_{10}) = \frac{4V}{\pi(2n-1)} \qquad &\to \qquad C_{10} = \frac{1}{c\gamma_n}\frac{4V}{\pi(2n-1)}. \end{aligned}$$

Consequently,

$$\begin{aligned} a_n(t) &= -\frac{1}{c^2\gamma_n^2}\frac{4k}{\pi(2n-1)} \cos c\gamma_n t + \frac{1}{c\gamma_n}\frac{4V}{\pi(2n-1)} \sin c\gamma_n t + \frac{1}{c^2\gamma_n^2}\frac{4k}{\pi(2n-1)} \\ &= \frac{4}{c^2\gamma_n^2\pi(2n-1)}(-k\cos c\gamma_n t + Vc\gamma_n \sin c\gamma_n t + k) \\ &= \frac{4}{c^2\left[\frac{\pi}{2l}(2n-1)\right]^2 \pi(2n-1)} \left\{-k\cos\left[\frac{\pi}{2l}(2n-1)ct\right] + Vc\left[\frac{\pi}{2l}(2n-1)\right]\sin\left[\frac{\pi}{2l}(2n-1)ct\right] + k\right\} \\ &= \frac{16l^2}{c^2\pi^3(2n-1)^3} \left\{-k\cos\left[\frac{\pi}{2l}(2n-1)ct\right] + Vc\left[\frac{\pi}{2l}(2n-1)\right]\sin\left[\frac{\pi}{2l}(2n-1)ct\right] + k\right\}. \end{aligned}$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{16l^2}{c^2\pi^3(2n-1)^3} \left\{-k\cos\left[\frac{\pi}{2l}(2n-1)ct\right] + Vc\left[\frac{\pi}{2l}(2n-1)\right]\sin\left[\frac{\pi}{2l}(2n-1)ct\right] + k\right\} \sin\left[\frac{\pi}{2l}(2n-1)x\right], \quad 0 < x \leq l.$$

Note that the solution above does not satisfy $u_{tt} - c^2 u_{xx} = k$ at $x = 0$ because it satisfies $u(0,t) = 0$ there. It converges to the solution very slowly.

Method 2 - Without Using Term-by-Term Differentiation

The eigenfunctions of the Helmholtz equation are known to form a complete set, so all of the functions in the PDE can be expanded in terms of them.

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} A_n(t)\phi_n(x) \quad \rightarrow \quad u\phi_m = \sum_{n=1}^{\infty} A_n\phi_n\phi_m \quad \rightarrow \quad \int_0^l u\phi_n\,dx = A_n\int_0^l \phi_n^2\,dx = A_n\cdot\frac{l}{2} \\
\frac{\partial^2 u}{\partial t^2} &= \sum_{n=1}^{\infty} B_n(t)\phi_n(x) \quad \rightarrow \quad \frac{\partial^2 u}{\partial t^2}\phi_m = \sum_{n=1}^{\infty} B_n\phi_n\phi_m \quad \rightarrow \quad \int_0^l \frac{\partial^2 u}{\partial t^2}\phi_n\,dx = B_n\int_0^l \phi_n^2\,dx = B_n\cdot\frac{l}{2} \\
k &= \sum_{n=1}^{\infty} D_n(t)\phi_n(x) \quad \rightarrow \quad k\phi_m = \sum_{n=1}^{\infty} D_n\phi_n\phi_m \quad \rightarrow \quad k\int_0^l \phi_n\,dx = D_n\int_0^l \phi_n^2\,dx = D_n\cdot\frac{l}{2} \\
\frac{\partial^2 u}{\partial x^2} &= \sum_{n=1}^{\infty} E_n(t)\phi_n(x) \quad \rightarrow \quad \frac{\partial^2 u}{\partial x^2}\phi_m = \sum_{n=1}^{\infty} E_n\phi_n\phi_m \quad \rightarrow \quad \int_0^l \frac{\partial^2 u}{\partial x^2}\phi_n\,dx = E_n\int_0^l \phi_n^2\,dx = E_n\cdot\frac{l}{2}
\end{aligned}$$

It should be emphasized that these are generalized Fourier series expansions for the functions, not product solutions that come about from using the method of separation of variables. Solve the latter equations for the generalized Fourier coefficients.

$$\begin{aligned}
A_n(t) &= \frac{2}{l}\int_0^l u\phi_n\,dx \\
B_n(t) &= \frac{2}{l}\int_0^l \frac{\partial^2 u}{\partial t^2}\phi_n\,dx = \frac{d^2}{dt^2}\left(\frac{2}{l}\int_0^l u\phi_n\,dx\right) = \frac{d^2A_n}{dt^2} \\
D_n(t) &= \frac{2k}{l}\int_0^l \phi_n\,dx = \frac{2k}{l}\cdot\frac{2l}{\pi(2n-1)} = \frac{4k}{\pi(2n-1)} \\
E_n(t) &= \frac{2}{l}\int_0^l \frac{\partial^2 u}{\partial x^2}\phi_n\,dx = \frac{2}{l}\Bigg(\underbrace{\frac{\partial u}{\partial x}\phi_n\Big|_0^l}_{=\,0} - \int_0^l \frac{\partial u}{\partial x}\frac{d\phi_n}{dx}\,dx\Bigg) = -\frac{2}{l}\int_0^l \frac{\partial u}{\partial x}\left[\frac{\pi}{2l}(2n-1)\right]\cos\left[\frac{\pi}{2l}(2n-1)x\right]dx
\end{aligned}$$

Apply integration by parts once more in order to write E_n in terms of A_n.

$$\begin{aligned}
E_n(t) &= -\frac{2}{l}\left[\frac{\pi}{2l}(2n-1)\right]\int_0^l \frac{\partial u}{\partial x}\cos\left[\frac{\pi}{2l}(2n-1)x\right]dx \\
&= -\frac{2}{l}\left[\frac{\pi}{2l}(2n-1)\right]\Bigg\{\underbrace{u\cos\left[\frac{\pi}{2l}(2n-1)x\right]\Big|_0^l}_{=\,0} - \int_0^l u\left[-\frac{\pi}{2l}(2n-1)\right]\sin\left[\frac{\pi}{2l}(2n-1)x\right]dx\Bigg\} \\
&= -\left[\frac{\pi}{2l}(2n-1)\right]^2\left\{\frac{2}{l}\int_0^l u\sin\left[\frac{\pi}{2l}(2n-1)x\right]dx\right\} \\
&= -\left[\frac{\pi}{2l}(2n-1)\right]^2 A_n
\end{aligned}$$

Now that the coefficients are known, substitute the eigenfunction expansions into the PDE.

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + k \\ \sum_{n=1}^{\infty} B_n(t)\phi_n(x) &= c^2 \sum_{n=1}^{\infty} E_n(t)\phi_n(x) + \sum_{n=1}^{\infty} D_n(t)\phi_n(x) \\ \sum_{n=1}^{\infty} B_n(t)\phi_n(x) &= \sum_{n=1}^{\infty} [c^2 E_n(t) + D_n(t)]\phi_n(x) \end{aligned}$$

Thus,

$$B_n(t) = c^2 E_n(t) + D_n(t).$$

Substitute the formulas for B_n, E_n, and D_n to obtain an ODE for A_n exclusively.

$$\frac{d^2 A_n}{dt^2} = -c^2 \left[\frac{\pi}{2l}(2n-1)\right]^2 A_n + \frac{4k}{\pi(2n-1)}$$

Bring the term with A_n to the left side.

$$\frac{d^2 A_n}{dt^2} + c^2 \left[\frac{\pi}{2l}(2n-1)\right]^2 A_n = \frac{4k}{\pi(2n-1)}$$

This is the same ODE that was obtained for a_n in Method 1, since the term in square brackets is γ_n. The initial conditions are also the same as before, so $A_n(t) = a_n(t)$.

$$A_n(t) = \frac{16l^2}{c^2\pi^3(2n-1)^3} \left\{-k\cos\left[\frac{\pi}{2l}(2n-1)ct\right] + Vc\left[\frac{\pi}{2l}(2n-1)\right]\sin\left[\frac{\pi}{2l}(2n-1)ct\right] + k\right\}.$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{16l^2}{c^2\pi^3(2n-1)^3} \left\{-k\cos\left[\frac{\pi}{2l}(2n-1)ct\right] + Vc\left[\frac{\pi}{2l}(2n-1)\right]\sin\left[\frac{\pi}{2l}(2n-1)ct\right] + k\right\} \sin\left[\frac{\pi}{2l}(2n-1)x\right], \quad 0 < x \le l.$$

Note that the solution above does not satisfy $u_{tt} - c^2 u_{xx} = k$ at $x = 0$ because it satisfies $u(0,t) = 0$ there. It converges to the solution very slowly.

Method 3 - Mr. Strauss's Way

Alternatively, because the source function and initial conditions are independent of time, another way to solve the PDE is to make the change of variables $v(x,t) = u(x,t) - r(x)$, or $u(x,t) = v(x,t) + r(x)$.

$$u_{tt} = c^2 u_{xx} + k \quad \rightarrow \quad v_{tt} = c^2\left(v_{xx} + \frac{d^2r}{dx^2}\right) + k$$
$$v_{tt} = c^2 v_{xx} + c^2\frac{d^2r}{dx^2} + k$$

If we set

$$c^2\frac{d^2r}{dx^2} + k = 0,$$

then the previous equation becomes

$$v_{tt} = c^2 v_{xx}.$$

Let r satisfy the same boundary conditions as u, that is,

$$r(0) = 0$$
$$r'(l) = 0.$$

The general solution for r is obtained by integrating both sides with respect to x twice.

$$\frac{d^2r}{dx^2} = -\frac{k}{c^2}$$

After the first integration, we have

$$\frac{dr}{dx} = -\frac{k}{c^2}x + C_{11}.$$

Apply the boundary condition at $x = l$ here to determine C_{11}.

$$\frac{dr}{dx}(l) = -\frac{k}{c^2}l + C_{11} = 0 \quad \rightarrow \quad C_{11} = \frac{k}{c^2}l$$

So then

$$\frac{dr}{dx} = -\frac{k}{c^2}x + \frac{k}{c^2}l.$$

Integrate both sides with respect to x once more.

$$r(x) = -\frac{k}{2c^2}x^2 + \frac{k}{c^2}lx + C_{12}.$$

Apply the boundary condition at $x = 0$ here to determine C_{12}.

$$r(0) = C_{12} = 0$$

The time-independent solution is consequently

$$r(x) = -\frac{k}{2c^2}x^2 + \frac{k}{c^2}lx$$
$$= \frac{k}{2c^2}x(2l - x).$$

Use the initial and boundary conditions for u to obtain those for v.

$$\begin{aligned}
u(0,t) &= v(0,t) + r(0) = v(0,t) + 0 = 0 && \rightarrow \quad v(0,t) = 0 \\
u_x(l,t) &= v_x(l,t) + r'(l) = v_x(l,t) + 0 = 0 && \rightarrow \quad v_x(l,t) = 0 \\
u(x,0) &= v(x,0) + r(x) = 0 && \rightarrow \quad v(x,0) = -r(x) = \frac{k}{2c^2}(x^2 - 2lx) \\
u_t(x,0) &= v_t(x,0) = V && \rightarrow \quad v_t(x,0) = V
\end{aligned}$$

Because the PDE for v and its associated boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve the equation. Assume a product solution of the form $v = X(x)T(t)$ and substitute it into the PDE

$$v_{tt} = c^2 v_{xx} \quad \rightarrow \quad XT'' = c^2 X''T$$

and the boundary conditions.

$$\begin{aligned}
v(0,t) &= 0 \quad \rightarrow \quad X(0)T(t) = 0 \quad \rightarrow \quad X(0) = 0 \\
v_x(l,t) &= 0 \quad \rightarrow \quad X'(l)T(t) = 0 \quad \rightarrow \quad X'(l) = 0
\end{aligned}$$

Now separate variables in the PDE: bring all constants and functions of t to the left and all functions of x to the right side.

$$\frac{T''}{c^2 T} = \frac{X''}{X}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda$$

Multiplying both sides of the second equation by X, we get

$$X'' = \lambda X,$$

which is the same eigenvalue problem ϕ satisfies. Thus, $\lambda = \lambda_n = -\gamma_n^2$, where

$$\gamma_n = \frac{\pi}{2l}(2n-1), \quad n = 1, 2, \ldots.$$

The eigenfunctions associated with these eigenvalues for λ are

$$X(x) = C_6 \sin \gamma x \quad \rightarrow \quad X_n(x) = \sin\left[\frac{\pi}{2l}(2n-1)x\right], \quad n = 1, 2, \ldots.$$

The aim now is to solve the ODE for T. Multiply both sides of it by $c^2 T$.

$$T'' = -\gamma_n^2 c^2 T$$

The general solution can be written in terms of sine and cosine.

$$\begin{aligned}
T(t) &= C_{13} \cos c\gamma_n t + C_{14} \sin c\gamma_n t \\
&= C_{13} \cos\left[\frac{\pi}{2l}(2n-1)ct\right] + C_{14} \sin\left[\frac{\pi}{2l}(2n-1)ct\right]
\end{aligned}$$

According to the principle of superposition, the general solution to the PDE for v is a linear combination of the eigenfunctions over all the eigenvalues.

$$v(x,t) = \sum_{n=1}^{\infty} \left\{ F_n \cos\left[\frac{\pi}{2l}(2n-1)ct\right] + G_n \sin\left[\frac{\pi}{2l}(2n-1)ct\right] \right\} \sin\left[\frac{\pi}{2l}(2n-1)x\right]$$

Use the initial conditions for v to determine the coefficients, F_n and G_n. Start with $v(x,0) = -r(x)$.

$$v(x,0) = \sum_{n=1}^{\infty} F_n \sin\left[\frac{\pi}{2l}(2n-1)x\right] = \frac{k}{2c^2}(x^2 - 2lx)$$

To solve for F_n, multiply both sides by $\sin \gamma_m$

$$\sum_{n=1}^{\infty} F_n \sin\left[\frac{\pi}{2l}(2n-1)x\right] \sin\left[\frac{\pi}{2l}(2m-1)x\right] = \frac{k}{2c^2}(x^2 - 2lx) \sin\left[\frac{\pi}{2l}(2m-1)x\right]$$

and then integrate both sides with respect to x from 0 to l.

$$\int_0^l \sum_{n=1}^{\infty} F_n \sin\left[\frac{\pi}{2l}(2n-1)x\right] \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx = \int_0^l \frac{k}{2c^2}(x^2 - 2lx) \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty} F_n \int_0^l \sin\left[\frac{\pi}{2l}(2n-1)x\right] \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx = \frac{k}{2c^2} \int_0^l (x^2 - 2lx) \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx$$

Because the eigenfunctions are orthogonal, the integral on the left side is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$F_n \int_0^l \sin^2\left[\frac{\pi}{2l}(2n-1)x\right] dx = \frac{k}{2c^2} \int_0^l (x^2 - 2lx) \sin\left[\frac{\pi}{2l}(2n-1)x\right] dx$$

Evaluate the integrals.

$$F_n \cdot \frac{l}{2} = \frac{k}{2c^2}\left[-\frac{16l^3}{\pi^3(2n-1)^3}\right]$$

Multiply both sides by $2/l$ and simplify.

$$F_n = -\frac{16kl^2}{c^2\pi^3(2n-1)^3}$$

Now take a derivative of $v(x,t)$ with respect to t

$$v_t(x,t) = \sum_{n=1}^{\infty} \left[\frac{\pi}{2l}(2n-1)c\right] \left\{ -F_n \sin\left[\frac{\pi}{2l}(2n-1)ct\right] + G_n \cos\left[\frac{\pi}{2l}(2n-1)ct\right] \right\} \sin\left[\frac{\pi}{2l}(2n-1)x\right]$$

and use the other initial condition to find G_n.

$$v_t(x,t) = \sum_{n=1}^{\infty} \left[\frac{\pi}{2l}(2n-1)c\right] G_n \sin\left[\frac{\pi}{2l}(2n-1)x\right] = V$$

To solve for G_n, multiply both sides by $\sin \gamma_m$

$$\sum_{n=1}^{\infty} \left[\frac{\pi}{2l}(2n-1)c\right] G_n \sin\left[\frac{\pi}{2l}(2n-1)x\right] \sin\left[\frac{\pi}{2l}(2m-1)x\right] = V \sin\left[\frac{\pi}{2l}(2m-1)x\right]$$

and then integrate both sides with respect to x from 0 to l.

$$\int_0^l \sum_{n=1}^{\infty} \left[\frac{\pi}{2l}(2n-1)c\right] G_n \sin\left[\frac{\pi}{2l}(2n-1)x\right] \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx = \int_0^l V \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty} \left[\frac{\pi}{2l}(2n-1)c\right] G_n \int_0^l \sin\left[\frac{\pi}{2l}(2n-1)x\right] \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx = V \int_0^l \sin\left[\frac{\pi}{2l}(2m-1)x\right] dx$$

Because the eigenfunctions are orthogonal, the integral on the left side is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$\left[\frac{\pi}{2l}(2n-1)c\right] G_n \int_0^l \sin^2\left[\frac{\pi}{2l}(2n-1)x\right] dx = V \int_0^l \sin\left[\frac{\pi}{2l}(2n-1)x\right] dx$$

Evaluate the integrals.

$$\left[\frac{\pi}{2l}(2n-1)c\right] G_n \cdot \frac{l}{2} = V \left[\frac{2l}{\pi(2n-1)}\right]$$

Solve for G_n.

$$G_n = \frac{8Vl}{c\pi^2(2n-1)^2}$$

The solution for v is then

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty} \left\{-\frac{16kl^2}{c^2\pi^3(2n-1)^3} \cos\left[\frac{\pi}{2l}(2n-1)ct\right] + \frac{8Vl}{c\pi^2(2n-1)^2} \sin\left[\frac{\pi}{2l}(2n-1)ct\right]\right\} \sin\left[\frac{\pi}{2l}(2n-1)x\right] \\ &= \sum_{n=1}^{\infty} \frac{8l}{c\pi^2(2n-1)^2} \left\{-\frac{2kl}{c\pi(2n-1)} \cos\left[\frac{\pi}{2l}(2n-1)ct\right] + V \sin\left[\frac{\pi}{2l}(2n-1)ct\right]\right\} \sin\left[\frac{\pi}{2l}(2n-1)x\right] \\ &= \frac{8l}{c\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \left\{V \sin\left[\frac{\pi}{2l}(2n-1)ct\right] - \frac{2kl}{c\pi(2n-1)} \cos\left[\frac{\pi}{2l}(2n-1)ct\right]\right\} \sin\left[\frac{\pi}{2l}(2n-1)x\right]. \end{aligned}$$

Therefore, since $u(x,t) = v(x,t) + r(x)$,

$$\begin{aligned} u(x,t) &= \frac{k}{2c^2} x(2l-x) \\ &\quad + \frac{8l}{c\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \left\{V \sin\left[\frac{\pi}{2l}(2n-1)ct\right] - \frac{2kl}{c\pi(2n-1)} \cos\left[\frac{\pi}{2l}(2n-1)ct\right]\right\} \sin\left[\frac{\pi}{2l}(2n-1)x\right], \\ &\qquad 0 \le x \le l. \end{aligned}$$

This solution converges much faster than the one obtained from using Method 1 and 2.