Exercise 7

Repeat Exercise 6 for the damped wave equation $u_{tt} = c^2 u_{xx} - r u_t + g(x)\sin \omega t$, where r is a positive constant.

Solution

The damped wave equation is subject to homogeneous initial and boundary conditions.

$$\begin{gathered} u_{tt} = c^2 u_{xx} - r u_t + g(x)\sin \omega t, \quad 0 < x < l, \quad t > 0 \\ u(0,t) = 0 \quad u(l,t) = 0 \\ u(x,0) = 0 \quad u_t(x,0) = 0. \end{gathered}$$

Since the PDE is linear and inhomogeneous, we choose to apply the method of eigenfunction expansion to solve it. Consider the eigenvalue problem of the differential operator involving the spatial variable x

$$\frac{d^2}{dx^2}\phi = \lambda\phi \tag{1}$$

with the same boundary conditions as u.

$$\begin{gathered} \phi(0) = 0 \\ \phi(l) = 0 \end{gathered}$$

Values of λ for which the boundary conditions are satisfied are known as the eigenvalues, and the nontrivial solutions associated with them are called the eigenfunctions. Equation (1) is known as the one-dimensional Helmholtz equation; the eigenfunctions for it are known to be orthogonal and form a complete set, which will prove useful later.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Suppose that λ is positive. Then equation (1) becomes

$$\frac{d^2\phi}{dx^2} = \mu^2\phi.$$

Its solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\phi(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} \phi(0) &= C_1 = 0 \\ \phi(l) &= C_1 \cosh \mu l + C_2 \sinh \mu l = 0 \end{aligned}$$

Since $C_1 = 0$, the second equation reduces to $C_2 \sinh \mu l = 0$. Hyperbolic sine is not oscillatory, so the only way this equation is satisfied is if $C_2 = 0$. The trivial solution is obtained, so there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Suppose that λ is zero. Then equation (1) becomes

$$\frac{d^2\phi}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\phi(x) = C_3 x + C_4$$

Apply the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} \phi(0) &= C_4 = 0 \\ \phi(l) &= C_3 l + C_4 = 0 \end{aligned}$$

Since $C_4 = 0$, the second equation reduces to $C_3 = 0$. The trivial solution is obtained, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Suppose that λ is negative. Then equation (1) becomes

$$\frac{d^2\phi}{dx^2} = -\gamma^2\phi.$$

Its solution can be written in terms of sine and cosine.

$$\phi(x) = C_5 \cos\gamma x + C_6 \sin\gamma x$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} \phi(0) &= C_5 = 0 \\ \phi(l) &= C_5 \cos\gamma l + C_6 \sin\gamma l = 0 \end{aligned}$$

Since $C_5 = 0$, the second equation reduces to $C_6 \sin\gamma l = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned} \sin\gamma l &= 0 \\ \gamma l &= n\pi, \quad n = 1, 2, \ldots \\ \gamma_n &= \frac{n\pi}{l}, \quad n = 1, 2, \ldots. \end{aligned}$$

The eigenfunctions associated with these eigenvalues for λ are

$$\phi(x) = C_6 \sin\gamma x \quad \rightarrow \quad \phi_n(x) = \sin\frac{n\pi x}{l}, \quad n = 1, 2, \ldots.$$

Method 1 - Using Term-by-Term Differentiation

The eigenfunctions of the Helmholtz equation form a complete set, so the unknown function u can be expanded in terms of them.

$$u(x,t)=\sum_{n=1}^{\infty}a_n(t)\sin\frac{n\pi x}{l}$$

To determine the generalized Fourier coefficients $a_n(t)$, substitute this expansion into the PDE.

$$u_{tt}=c^2u_{xx}-ru_t+g(x)\sin\omega t$$

$$\frac{\partial^2}{\partial t^2}\sum_{n=1}^{\infty}a_n(t)\sin\frac{n\pi x}{l}=c^2\frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty}a_n(t)\sin\frac{n\pi x}{l}-r\frac{\partial}{\partial t}\sum_{n=1}^{\infty}a_n(t)\sin\frac{n\pi x}{l}+g(x)\sin\omega t$$

Because u satisfies homogeneous boundary conditions and u, $\partial u/\partial x$, and $\partial u/\partial t$ are continuous (reasonable assumptions for the displacement of a homogeneous elastic string), the three series can in fact be differentiated term by term.

$$\sum_{n=1}^{\infty}\frac{d^2a_n}{dt^2}\sin\frac{n\pi x}{l}=c^2\sum_{n=1}^{\infty}a_n(t)\frac{d^2}{dx^2}\sin\frac{n\pi x}{l}-r\sum_{n=1}^{\infty}\frac{da_n}{dt}\sin\frac{n\pi x}{l}+g(x)\sin\omega t$$

The operator applied to the eigenfunction is equal to the eigenvalue times the eigenfunction.

$$\sum_{n=1}^{\infty}\frac{d^2a_n}{dt^2}\sin\frac{n\pi x}{l}=c^2\sum_{n=1}^{\infty}a_n(t)\lambda_n\sin\frac{n\pi x}{l}-r\sum_{n=1}^{\infty}\frac{da_n}{dt}\sin\frac{n\pi x}{l}+g(x)\sin\omega t$$

Bring all series to the left side and combine them.

$$\sum_{n=1}^{\infty}\left[\frac{d^2a_n}{dt^2}+r\frac{da_n}{dt}-c^2\lambda_na_n(t)\right]\sin\frac{n\pi x}{l}=g(x)\sin\omega t$$

The left side is essentially a Fourier sine series expansion of $g(x)\sin\omega t$. To solve for the term in square brackets, multiply both sides by $\sin(m\pi x/l)$, where m is an integer,

$$\sum_{n=1}^{\infty}\left[\frac{d^2a_n}{dt^2}+r\frac{da_n}{dt}-c^2\lambda_na_n(t)\right]\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}=g(x)\sin\frac{m\pi x}{l}\sin\omega t$$

and then integrate both sides with respect to x from 0 to l.

$$\int_0^l\sum_{n=1}^{\infty}\left[\frac{d^2a_n}{dt^2}+r\frac{da_n}{dt}-c^2\lambda_na_n(t)\right]\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx=\int_0^l g(x)\sin\frac{m\pi x}{l}\sin\omega t\,dx$$

$g(x)$ is assumed not to be orthogonal to $\sin(m\pi x/l)$ so that the integral on the right side is nonzero. Bring the functions of t in front of the integrals.

$$\sum_{n=1}^{\infty}\left[\frac{d^2a_n}{dt^2}+r\frac{da_n}{dt}-c^2\lambda_na_n(t)\right]\int_0^l\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx=\sin\omega t\int_0^l g(x)\sin\frac{m\pi x}{l}\,dx$$

Since the eigenfunctions are orthogonal, the integral on the left side is zero if $n\neq m$. As a result, every term in the infinite series vanishes except for one: $n=m$.

$$\left[\frac{d^2a_n}{dt^2}+r\frac{da_n}{dt}-c^2\lambda_na_n(t)\right]\int_0^l\sin^2\frac{n\pi x}{l}\,dx=\sin\omega t\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$

Evaluate the integral on the left side.

$$\left[\frac{d^2a_n}{dt^2} + r\frac{da_n}{dt} - c^2\lambda_n a_n(t)\right] \cdot \frac{l}{2} = \sin\omega t \int_0^l g(x) \sin\frac{n\pi x}{l}\,dx$$

Multiply both sides by $2/l$ and replace λ_n with $-(n\pi/l)^2$.

$$\frac{d^2a_n}{dt^2} + r\frac{da_n}{dt} + c^2\frac{n^2\pi^2}{l^2}a_n = \left[\frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx\right]\sin\omega t$$

With the help of the method of eigenfunction expansion, the PDE has been reduced to a second-order inhomogeneous ODE. Because the ODE is linear, the general solution is the sum of a complementary solution and a particular solution.

$$a_n = a_c + a_p$$

The complementary solution satisfies the associated homogeneous equation.

$$\frac{d^2a_c}{dt^2} + r\frac{da_c}{dt} + c^2\frac{n^2\pi^2}{l^2}a_c = 0$$

Since the coefficients are constant, the solution is of the form $a_c = e^{pt}$. Take two derivatives of it

$$a_c = e^{pt} \quad\rightarrow\quad \frac{da_c}{dt} = pe^{pt} \quad\rightarrow\quad \frac{d^2a_c}{dt^2} = p^2e^{pt}$$

and substitute these expressions into the equation to find p.

$$p^2e^{pt} + rpe^{pt} + c^2\frac{n^2\pi^2}{l^2}e^{pt} = 0$$

$$p^2 + rp + c^2\frac{n^2\pi^2}{l^2} = 0$$

$$p = \frac{-r \pm \sqrt{r^2 - 4c^2\frac{n^2\pi^2}{l^2}}}{2} = -\frac{r}{2} \pm \frac{1}{2}\sqrt{r^2 - \left(\frac{2\pi c}{l}n\right)^2}$$

Depending on how large the damping parameter r is, p can either be real or complex.

$$\text{Case I:} \quad r = \frac{2\pi c}{l}n \quad \text{or} \quad n = \frac{rl}{2\pi c} \quad\Rightarrow\quad p = -\frac{r}{2} \quad \text{(repeated)}$$

$$\text{Case II:} \quad r > \frac{2\pi c}{l}n \quad \text{or} \quad n < \frac{rl}{2\pi c} \quad\Rightarrow\quad p = -\frac{r}{2} \pm \frac{1}{2}\sqrt{r^2 - \left(\frac{2\pi c}{l}n\right)^2}$$

$$\text{Case III:} \quad r < \frac{2\pi c}{l}n \quad \text{or} \quad n > \frac{rl}{2\pi c} \quad\Rightarrow\quad p = -\frac{r}{2} \pm \frac{i}{2}\sqrt{\left(\frac{2\pi c}{l}n\right)^2 - r^2}$$

The solution thus has three possible forms.

$$\text{Case I:} \quad a_{c_1}(t) = \exp\left(-\frac{r}{2}t\right)(C_7 + C_8 t)$$

$$\text{Case II:} \quad a_{c_2}(t) = \exp\left(-\frac{r}{2}t\right)\left\{C_9\cosh\left[\frac{t}{2}\sqrt{r^2 - \left(\frac{2\pi c}{l}n\right)^2}\right] + C_{10}\sinh\left[\frac{t}{2}\sqrt{r^2 - \left(\frac{2\pi c}{l}n\right)^2}\right]\right\}$$

$$\text{Case III:} \quad a_{c_3}(t) = \exp\left(-\frac{r}{2}t\right)\left\{C_{11}\cos\left[\frac{t}{2}\sqrt{\left(\frac{2\pi c}{l}n\right)^2 - r^2}\right] + C_{12}\sin\left[\frac{t}{2}\sqrt{\left(\frac{2\pi c}{l}n\right)^2 - r^2}\right]\right\}$$

Note that in any case, because of $e^{-rt/2}$, $a_c(t)$ tends to zero as $t \to \infty$. Since the inhomogeneous term is sine and there are odd derivatives, the particular solution is of the form
$a_p = C_{13}\cos\omega t + C_{14}\sin\omega t$. Take two derivatives of it

$$a_p = C_{13}\cos\omega t \;+\; C_{14}\sin\omega t \quad \to \quad \frac{da_p}{dt} = \omega(-C_{13}\sin\omega t \;+\; C_{14}\cos\omega t)$$
$$\to \quad \frac{d^2a_p}{dt^2} = \omega^2(-C_{13}\cos\omega t \;-\; C_{14}\sin\omega t)$$

and substitute these expressions into the ODE to find C_{13} and C_{14}.

$$\frac{d^2a_p}{dt^2} + r\frac{da_p}{dt} + c^2\frac{n^2\pi^2}{l^2}a_p = \left[\frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx\right]\sin\omega t$$

$$\frac{c^2n^2\pi^2C_{13} + \omega l^2(C_{14}r - C_{13}\omega)}{l^2}\cos\omega t \;+\; \frac{c^2n^2\pi^2C_{14} - \omega l^2(C_{13}r + C_{14}\omega)}{l^2}\sin\omega t$$
$$= \left[\frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx\right]\sin\omega t$$

Match the coefficients of sine and cosine in order to obtain a system of equations for the constants.

$$\frac{c^2n^2\pi^2C_{13} + \omega l^2(C_{14}r - C_{13}\omega)}{l^2} = 0$$
$$\frac{c^2n^2\pi^2C_{14} - \omega l^2(C_{13}r + C_{14}\omega)}{l^2} = \frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$

Solving the system yields

$$C_{13} = -\frac{2r\omega l^3}{r^2\omega^2l^4 + (\omega^2l^2 - c^2n^2\pi^2)^2}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$
$$C_{14} = -\frac{2l(\omega^2l^2 - c^2n^2\pi^2)}{r^2\omega^2l^4 + (\omega^2l^2 - c^2n^2\pi^2)^2}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$$

Hence, the general solution a_{n_i} in each of the three cases is $(i = 1, 2, 3)$

$$a_{n_i}(t) = a_{c_i}(t) - \frac{2r\omega l^3\cos\omega t}{r^2\omega^2l^4 + (\omega^2l^2 - c^2n^2\pi^2)^2}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$
$$- \frac{2l(\omega^2l^2 - c^2n^2\pi^2)\sin\omega t}{r^2\omega^2l^4 + (\omega^2l^2 - c^2n^2\pi^2)^2}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$
$$= a_{c_i}(t) + \frac{2l[c^2n^2\pi^2\sin\omega t - \omega l^2(r\cos\omega t + \omega\sin\omega t)]}{r^2\omega^2l^4 + (\omega^2l^2 - c^2n^2\pi^2)^2}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$$

Use the initial conditions for u in combination with the eigenfunction expansion to determine those for a_n.

$$u(x,0) = \sum_{n=1}^{\infty} a_n(0)\sin\frac{n\pi x}{l} = 0 \qquad \Rightarrow \qquad a_n(0) = 0$$
$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{da_n}{dt}(0)\sin\frac{n\pi x}{l} = 0 \qquad \Rightarrow \qquad \frac{da_n}{dt}(0) = 0$$

Apply them both to obtain a system of equations for C_7, C_8, C_9, C_{10}, C_{11}, and C_{12}.

$$a_n(0) = 0 \quad \rightarrow \quad \begin{cases} a_{n_1}(0) = C_7 - \dfrac{2r\omega l^3}{r^2\omega^2 l^4 + (\omega^2 l^2 - c^2 n^2 \pi^2)^2} \displaystyle\int_0^l g(x) \sin \frac{n\pi x}{l}\, dx = 0 \\ a_{n_2}(0) = C_9 - \dfrac{2r\omega l^3}{r^2\omega^2 l^4 + (\omega^2 l^2 - c^2 n^2 \pi^2)^2} \displaystyle\int_0^l g(x) \sin \frac{n\pi x}{l}\, dx = 0 \\ a_{n_3}(0) = C_{11} - \dfrac{2r\omega l^3}{r^2\omega^2 l^4 + (\omega^2 l^2 - c^2 n^2 \pi^2)^2} \displaystyle\int_0^l g(x) \sin \frac{n\pi x}{l}\, dx = 0 \end{cases}$$

$$\frac{da_n}{dt}(0) = 0 \quad \rightarrow \quad \begin{cases} \dfrac{da_{n_1}}{dt}(0) = -\dfrac{r}{2}C_7 + C_8 - \dfrac{2\omega l(\omega^2 l^2 - c^2 n^2 \pi^2)}{r^2\omega^2 l^4 + (\omega^2 l^2 - c^2 n^2 \pi^2)^2} \displaystyle\int_0^l g(x) \sin \frac{n\pi x}{l}\, dx = 0 \\ \dfrac{da_{n_2}}{dt}(0) = -\dfrac{r}{2}C_9 + \dfrac{C_{10}}{2}\sqrt{r^2 - \left(\dfrac{2\pi c}{l}n\right)^2} - \dfrac{2\omega l(\omega^2 l^2 - c^2 n^2 \pi^2)}{r^2\omega^2 l^4 + (\omega^2 l^2 - c^2 n^2 \pi^2)^2} \displaystyle\int_0^l g(x) \sin \frac{n\pi x}{l}\, dx = 0 \\ \dfrac{da_{n_3}}{dt}(0) = -\dfrac{r}{2}C_{11} + \dfrac{C_{12}}{2}\sqrt{\left(\dfrac{2\pi c}{l}n\right)^2 - r^2} - \dfrac{2\omega l(\omega^2 l^2 - c^2 n^2 \pi^2)}{r^2\omega^2 l^4 + (\omega^2 l^2 - c^2 n^2 \pi^2)^2} \displaystyle\int_0^l g(x) \sin \frac{n\pi x}{l}\, dx = 0 \end{cases}$$

Solving this system yields the following.

$$C_7 = \frac{2r\omega l^3}{r^2\omega^2 l^4 + (\omega^2 l^2 - c^2 n^2 \pi^2)^2} \int_0^l g(x) \sin \frac{n\pi x}{l}\, dx \qquad C_8 = \frac{\omega l[2(\omega^2 l^2 - c^2 n^2 \pi^2) + r^2 l^2]}{r^2\omega^2 l^4 + (\omega^2 l^2 - c^2 n^2 \pi^2)^2} \int_0^l g(x) \sin \frac{n\pi x}{l}\, dx$$

$$C_9 = \frac{2r\omega l^3}{r^2\omega^2 l^4 + (\omega^2 l^2 - c^2 n^2 \pi^2)^2} \int_0^l g(x) \sin \frac{n\pi x}{l}\, dx \qquad C_{10} = \frac{2\omega l^2[2(\omega^2 l^2 - c^2 n^2 \pi^2) + r^2 l^2]}{[r^2\omega^2 l^4 + (\omega^2 l^2 - c^2 n^2 \pi^2)^2]\sqrt{r^2 l^2 - 4c^2 n^2 \pi^2}} \int_0^l g(x) \sin \frac{n\pi x}{l}\, dx$$

$$C_{11} = \frac{2r\omega l^3}{r^2\omega^2 l^4 + (\omega^2 l^2 - c^2 n^2 \pi^2)^2} \int_0^l g(x) \sin \frac{n\pi x}{l}\, dx \qquad C_{12} = \frac{2\omega l^2[2(\omega^2 l^2 - c^2 n^2 \pi^2) + r^2 l^2]}{[r^2\omega^2 l^4 + (\omega^2 l^2 - c^2 n^2 \pi^2)^2]\sqrt{4c^2 n^2 \pi^2 - r^2 l^2}} \int_0^l g(x) \sin \frac{n\pi x}{l}\, dx$$

The solution for u is known now that all the constants are determined. Depending on the magnitude of the ratio $rl/(2\pi c)$ and whether it is a positive integer or not, $a_{n_1}(t)$ and $a_{n_2}(t)$ may or may not be a part of the solution. Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{l} = \begin{cases} \sum_{n=1}^{\infty} a_{n_3}(t) \sin \frac{n\pi x}{l} & \text{if } \frac{rl}{2\pi c} < 1 \\ a_{n_1}(t) \sin \frac{n\pi x}{l}\Big|_{n=\frac{rl}{2\pi c}=1} + \sum_{n=2}^{\infty} a_{n_3}(t) \sin \frac{n\pi x}{l} & \text{if } \frac{rl}{2\pi c} = 1 \\ \sum_{1 \le n < \frac{rl}{2\pi c}}^{\infty} a_{n_2}(t) \sin \frac{n\pi x}{l} + \sum_{\frac{rl}{2\pi c} < n < \infty}^{\infty} a_{n_3}(t) \sin \frac{n\pi x}{l} & \text{if } \frac{rl}{2\pi c} > 1 \text{ and } \frac{rl}{2\pi c} \notin \mathbb{Z}^+ \\ \sum_{1 \le n < \frac{rl}{2\pi c}}^{\infty} a_{n_2}(t) \sin \frac{n\pi x}{l} + a_{n_1}(t) \sin \frac{n\pi x}{l}\Big|_{n=\frac{rl}{2\pi c}} + \sum_{\frac{rl}{2\pi c} < n < \infty}^{\infty} a_{n_3}(t) \sin \frac{n\pi x}{l} & \text{if } \frac{rl}{2\pi c} > 1 \text{ and } \frac{rl}{2\pi c} \in \mathbb{Z}^+ \end{cases}.$$

Plugging in all the constants and simplifying, the explicit formulas for $a_{n_1}(t)$, $a_{n_2}(t)$, and $a_{n_3}(t)$ are therefore

$$a_{n_1}(t) = \frac{e^{-\frac{rt}{2}}[r\omega(rt+4) + 4t\omega^3] - 4r\omega\cos\omega t - (4\omega^2 - r^2)\sin\omega t}{(4\omega^2 + r^2)^2} \times \frac{8}{l}\int_0^l g(x)\sin\frac{rx}{2c}\,dx$$

$$a_{n_2}(t) = \left\{\frac{le^{-\frac{rt}{2}}\left\{r\omega l\sqrt{r^2l^2 - 4c^2n^2\pi^2}\cosh\left(\frac{t}{2l}\sqrt{r^2l^2 - 4c^2n^2\pi^2}\right) + \omega[2(\omega^2l^2 - c^2n^2\pi^2) + r^2l^2]\sinh\left(\frac{t}{2l}\sqrt{r^2l^2 - 4c^2n^2\pi^2}\right)\right\}}{[r^2\omega^2l^4 + (\omega^2l^2 - c^2n^2\pi^2)^2]\sqrt{r^2l^2 - 4c^2n^2\pi^2}} \right.$$
$$\left. + \frac{c^2n^2\pi^2\sin\omega t - \omega l^2(r\cos\omega t + \omega\sin\omega t)}{r^2\omega^2l^4 + (\omega^2l^2 - c^2n^2\pi^2)^2}\right\} \times 2l\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$

$$a_{n_3}(t) = \left\{\frac{le^{-\frac{rt}{2}}\left\{r\omega l\sqrt{4c^2n^2\pi^2 - r^2l^2}\cos\left(\frac{t}{2l}\sqrt{4c^2n^2\pi^2 - r^2l^2}\right) + \omega[2(\omega^2l^2 - c^2n^2\pi^2) + r^2l^2]\sin\left(\frac{t}{2l}\sqrt{4c^2n^2\pi^2 - r^2l^2}\right)\right\}}{[r^2\omega^2l^4 + (\omega^2l^2 - c^2n^2\pi^2)^2]\sqrt{4c^2n^2\pi^2 - r^2l^2}} \right.$$
$$\left. + \frac{c^2n^2\pi^2\sin\omega t - \omega l^2(r\cos\omega t + \omega\sin\omega t)}{r^2\omega^2l^4 + (\omega^2l^2 - c^2n^2\pi^2)^2}\right\} \times 2l\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$$

No resonance occurs because the solutions do not blow up or grow in time. Note that $r^2l^2 \neq 4c^2n^2\pi^2$ in $a_{n_2}(t)$ and $a_{n_3}(t)$.

Method 2 - Without Using Term-by-Term Differentiation

The eigenfunctions of the Helmholtz equation are known to form a complete set, so all of the functions in the PDE can be expanded in terms of them.

$$u(x,t)=\sum_{n=1}^{\infty}A_n(t)\phi_n(x) \quad\rightarrow\quad u\phi_m=\sum_{n=1}^{\infty}A_n\phi_n\phi_m \quad\rightarrow\quad \int_0^l u\phi_n\,dx=A_n\int_0^l \phi_n^2\,dx=A_n\cdot\frac{l}{2}$$

$$\frac{\partial^2 u}{\partial t^2}=\sum_{n=1}^{\infty}B_n(t)\phi_n(x) \quad\rightarrow\quad \frac{\partial^2 u}{\partial t^2}\phi_m=\sum_{n=1}^{\infty}B_n\phi_n\phi_m \quad\rightarrow\quad \int_0^l \frac{\partial^2 u}{\partial t^2}\phi_n\,dx=B_n\int_0^l \phi_n^2\,dx=B_n\cdot\frac{l}{2}$$

$$g(x)\sin\omega t=\sum_{n=1}^{\infty}D_n(t)\phi_n(x) \quad\rightarrow\quad g(x)\sin\omega t\phi_m=\sum_{n=1}^{\infty}D_n\phi_n\phi_m \quad\rightarrow\quad \sin\omega t\int_0^l g\phi_n\,dx=D_n\int_0^l \phi_n^2\,dx=D_n\cdot\frac{l}{2}$$

$$\frac{\partial^2 u}{\partial x^2}=\sum_{n=1}^{\infty}E_n(t)\phi_n(x) \quad\rightarrow\quad \frac{\partial^2 u}{\partial x^2}\phi_m=\sum_{n=1}^{\infty}E_n\phi_n\phi_m \quad\rightarrow\quad \int_0^l \frac{\partial^2 u}{\partial x^2}\phi_n\,dx=E_n\int_0^l \phi_n^2\,dx=E_n\cdot\frac{l}{2}$$

$$\frac{\partial u}{\partial t}=\sum_{n=1}^{\infty}F_n(t)\phi_n(x) \quad\rightarrow\quad \frac{\partial u}{\partial t}\phi_m=\sum_{n=1}^{\infty}F_n\phi_n\phi_m \quad\rightarrow\quad \int_0^l \frac{\partial u}{\partial t}\phi_n\,dx=F_n\int_0^l \phi_n^2\,dx=F_n\cdot\frac{l}{2}$$

It should be emphasized that these are generalized Fourier series expansions for the functions, not product solutions that come about from using the method of separation of variables. Solve the latter equations for the generalized Fourier coefficients.

$$\begin{aligned}
A_n(t)&=\frac{2}{l}\int_0^l u\phi_n\,dx\\
B_n(t)&=\frac{2}{l}\int_0^l \frac{\partial^2 u}{\partial t^2}\phi_n\,dx=\frac{d^2}{dt^2}\left(\frac{2}{l}\int_0^l u\phi_n\,dx\right)=\frac{d^2A_n}{dt^2}\\
D_n(t)&=\frac{2\sin\omega t}{l}\int_0^l g(x)\phi_n\,dx\\
E_n(t)&=\frac{2}{l}\int_0^l \frac{\partial^2 u}{\partial x^2}\phi_n\,dx=\frac{2}{l}\left(\underbrace{\frac{\partial u}{\partial x}\phi_n\bigg|_0^l}_{=0}-\int_0^l \frac{\partial u}{\partial x}\frac{d\phi_n}{dx}\,dx\right)=-\frac{2n\pi}{l^2}\int_0^l \frac{\partial u}{\partial x}\cos\frac{n\pi x}{l}\,dx\\
F_n(t)&=\frac{2}{l}\int_0^l \frac{\partial u}{\partial t}\phi_n\,dx=\frac{d}{dt}\left(\frac{2}{l}\int_0^l u\phi_n\,dx\right)=\frac{dA_n}{dt}
\end{aligned}$$

Apply integration by parts once more in order to write E_n in terms of A_n.

$$\begin{aligned}
E_n(t)&=-\frac{2n\pi}{l^2}\left[\underbrace{u\cos\frac{n\pi x}{l}\bigg|_0^l}_{=0}-\int_0^l u\left(-\frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\,dx\right]\\
&=-\frac{n^2\pi^2}{l^2}\left(\frac{2}{l}\int_0^l u\sin\frac{n\pi x}{l}\,dx\right)\\
&=-\frac{n^2\pi^2}{l^2}A_n
\end{aligned}$$

Now that the coefficients are known, substitute the eigenfunction expansions into the PDE.

$$\begin{aligned} u_{tt} &= c^2 u_{xx} - r u_t + g(x)\sin\omega t \\ \sum_{n=1}^{\infty} B_n(t)\phi_n(x) &= c^2 \sum_{n=1}^{\infty} E_n(t)\phi_n(x) - r\sum_{n=1}^{\infty} F_n(t)\phi_n(x) + \sum_{n=1}^{\infty} D_n(t)\phi_n(x) \\ \sum_{n=1}^{\infty} B_n(t)\phi_n(x) &= \sum_{n=1}^{\infty} [c^2 E_n(t) - rF_n(t) + D_n(t)]\phi_n(x) \end{aligned}$$

Thus,

$$B_n(t) = c^2 E_n(t) - rF_n(t) + D_n(t).$$

Substitute the formulas for B_n, E_n, and D_n to obtain an ODE for A_n exclusively.

$$\frac{d^2 A_n}{dt^2} = -c^2 \frac{n^2\pi^2}{l^2} A_n - r\frac{dA_n}{dt} + \frac{2\sin\omega t}{l}\int_0^l g(x)\phi_n\, dx$$

Bring the terms with A_n to the left side and replace ϕ_n with $\sin(n\pi x/l)$.

$$\frac{d^2 A_n}{dt^2} + r\frac{dA_n}{dt} + c^2 \frac{n^2\pi^2}{l^2} A_n = \left[\frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\, dx\right]\sin\omega t$$

This is the same ODE that was obtained for a_n in Method 1. The initial conditions are also the same as before, so $A_n(t) = a_n(t)$. Therefore, the same function for u is obtained.