Exercise 9

Use the method of subtraction to solve $u_{tt} = 9u_{xx}$ for $0 \leq x \leq 1 = l$, with $u(0,t) = h$, $u(1,t) = k$, where h and k are given constants, and $u(x,0) = 0$, $u_t(x,0) = 0$.

Solution

Since the boundary conditions are constants, they can be made homogeneous by making the substitution $u(x,t) = v(x,t) + f(x)$ in the PDE.

$$\begin{aligned} \frac{\partial^2}{\partial t^2}[v(x,t) + f(x)] &= 9\frac{\partial^2}{\partial x^2}[v(x,t) + f(x)] \\ v_{tt} &= 9\left(v_{xx} + \frac{d^2 f}{dx^2}\right) \\ v_{tt} &= 9v_{xx} + 9\frac{d^2 f}{dx^2} \end{aligned}$$

If we set

$$9\frac{d^2 f}{dx^2} = 0,$$

then the previous equation reduces to

$$v_{tt} = 9v_{xx}.$$

The ODE for f can be solved by integrating both sides twice with respect to x.

$$\frac{d^2 f}{dx^2} = 0 \quad \rightarrow \quad \frac{df}{dx} = C_1 \quad \rightarrow \quad f(x) = C_1 x + C_2$$

Let the boundary conditions for f be the same as those for u: $f(0) = h$ and $f(1) = k$. Apply them both to determine C_1 and C_2.

$$\begin{aligned} f(0) &= C_2 = h \\ f(1) &= C_1 + C_2 = k \end{aligned}$$

Solving the second equation yields $C_1 = k - h$, so

$$f(x) = (k - h)x + h.$$

Before solving the PDE for v, determine the initial and boundary conditions associated with it.

$$\begin{aligned} v(0,t) &= u(0,t) - f(0) = h - h = 0 \\ v(1,t) &= u(1,t) - f(1) = k - k = 0 \\ v(x,0) &= u(x,0) - f(x) = 0 - f(x) = (h - k)x - h \\ v_t(x,0) &= u_t(x,0) = 0 \end{aligned}$$

The PDE and its boundary conditions are linear and homogeneous, so the method of separation of variables can be applied to solve it. Assume a product solution $v = X(x)T(t)$ and substitute it into the PDE

$$v_{tt} = 9v_{xx} \quad \rightarrow \quad XT'' = 9X''T$$

and the boundary conditions.

$$\begin{aligned} v(0,t) &= 0 & \rightarrow && X(0)T(t) &= 0 & \rightarrow && X(0) &= 0 \\ v(1,t) &= 0 & \rightarrow && X(1)T(t) &= 0 & \rightarrow && X(1) &= 0 \end{aligned}$$

Now separate variables in the PDE: bring all constants and functions of t to the left side and all functions of x to the right side.

$$\frac{T''}{9T} = \frac{X''}{X}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{T''}{9T} = \frac{X''}{X} = \lambda$$

Values of λ for which the boundary conditions are satisfied are known as the eigenvalues, and the nontrivial solutions associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Suppose that λ is positive. Then the ODE for x becomes

$$\frac{X''}{X} = \mu^2.$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

Its solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_3 \cosh \mu x + C_4 \sinh \mu x$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_3 = 0 \\ X(1) &= C_3 \cosh \mu + C_4 \sinh \mu = 0 \end{aligned}$$

Since $C_3 = 0$, the second equation reduces to $C_4 \sinh \mu = 0$. Hyperbolic sine is not oscillatory, so the only way this equation is satisfied is if $C_4 = 0$. The trivial solution is obtained, so there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Suppose that λ is zero. Then the ODE for X becomes

$$\frac{X''}{X} = 0.$$

Multiply both sides by X.

$$X'' = 0$$

The general solution is obtained by integrating both sides with respect to x twice.

$$X(x) = C_5 x + C_6$$

Apply the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_6 = 0 \\ X(1) &= C_5 + C_6 = 0 \end{aligned}$$

Since $C_6 = 0$, the second equation reduces to $C_5 = 0$. The trivial solution is obtained, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Suppose that λ is negative. Then the ODE for X becomes

$$\frac{X''}{X} = -\gamma^2.$$

Multiply both sides by X.

$$X'' = -\gamma^2 X$$

Its solution can be written in terms of sine and cosine.

$$X(x) = C_7 \cos\gamma x + C_8 \sin\gamma x$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_7 = 0 \\ X(1) &= C_7 \cos\gamma + C_8 \sin\gamma = 0 \end{aligned}$$

Since $C_7 = 0$, the second equation reduces to $C_8 \sin\gamma l = 0$. To avoid getting the trivial solution, we insist that $C_8 \neq 0$. Then

$$\begin{aligned} \sin\gamma &= 0 \\ \gamma_n &= n\pi, \quad n = 1, 2, \ldots. \end{aligned}$$

The eigenfunctions associated with these eigenvalues for λ are

$$X(x) = C_6 \sin\gamma x \quad \rightarrow \quad X_n(x) = \sin n\pi x, \quad n = 1, 2, \ldots.$$

Now the related ODE for T will be solved with this value of λ.

$$\frac{T''}{9T} = -n^2\pi^2$$

Multiply both sides by $9T$.

$$T'' = -9n^2\pi^2 T$$

The general solution can be written in terms of sine and cosine.

$$T(t) = C_9 \cos 3n\pi t + C_{10} \sin 3n\pi t$$

According to the principle of superposition, the general solution for v is a linear combination of the eigenfunctions over all the eigenvalues.

$$v(x,t) = \sum_{n=1}^{\infty} (A_n \cos 3n\pi t + B_n \sin 3n\pi t) \sin n\pi x$$

Apply the first initial condition for v now to determine one of the coefficients.

$$v(x,0) = \sum_{n=1}^{\infty} A_n \sin n\pi x = (h-k)x - h$$

To solve for A_n, multiply both sides by $\sin m\pi x$, where m is an integer,

$$\sum_{n=1}^{\infty} A_n \sin n\pi x \sin m\pi x = [(h-k)x - h]\sin m\pi x$$

and then integrate both sides with respect to x from 0 to 1.

$$\int_0^1 \sum_{n=1}^{\infty} A_n \sin n\pi x \sin m\pi x\, dx = \int_0^1 [(h-k)x - h]\sin m\pi x\, dx$$

Bring the constants in front of the integral on the left side.

$$\sum_{n=1}^{\infty} A_n \int_0^1 \sin n\pi x \sin m\pi x\, dx = \int_0^1 [(h-k)x - h]\sin m\pi x\, dx$$

Because the eigenfunctions are orthogonal, the integral on the left side is zero when $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$A_n \int_0^1 \sin^2 n\pi x\, dx = \int_0^1 [(h-k)x - h]\sin n\pi x\, dx$$

Evaluate the integrals.

$$A_n \cdot \frac{1}{2} = \frac{k(-1)^n - h}{n\pi}$$

Multiply both sides by 2 to solve for A_n.

$$A_n = \frac{2}{n\pi}[k(-1)^n - h]$$

Now take a derivative of v

$$v_t(x,t) = \sum_{n=1}^{\infty} 3n\pi(-A_n \sin 3n\pi t + B_n \cos 3n\pi t)\sin n\pi x$$

and use the second initial condition to determine B_n.

$$v_t(x,0) = \sum_{n=1}^{\infty} 3n\pi(B_n)\sin n\pi x = 0 \quad \rightarrow \quad B_n = 0$$

So then

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty} \frac{2}{n\pi}[k(-1)^n - h]\cos 3n\pi t \sin n\pi x \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{k(-1)^n - h}{n}\cos 3n\pi t \sin n\pi x. \end{aligned}$$

Therefore, since $u(x,t) = v(x,t) + f(x)$,

$$u(x,t) = (k-h)x + h + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{k(-1)^n - h}{n}\cos 3n\pi t \sin n\pi x, \quad 0 \le x \le 1.$$