Exercise 10

Find the temperature of a metal rod that is in the shape of a solid circular cone with cross-sectional area $A(x) = b(1 - x/l)^2$ for $0 \le x \le l$, where b is a constant. Assume that the rod is made of a uniform material, is insulated on its sides, is maintained at zero temperature on its flat end ($x = 0$), and has an unspecified initial temperature distribution $\phi(x)$. Assume that the temperature is independent of y and z. [*Hint:* Derive the PDE $(1 - x/l)^2 u_t = k\{(1 - x/l)^2 u_x\}_x$. Separate variables $u = T(t)X(x)$ and then substitute $v(x) = (1 - x/l)X(x)$.]

Solution

The law of conservation of energy states that energy is neither created nor destroyed. If some amount of thermal energy enters the left side of a rod at $x = a$, then that same amount must exit the right side of it at $x = b$ for the temperature to remain the same. If more (less) thermal energy enters at $x = a$ than exits at $x = b$, then the amount of thermal energy in the rod will change, leading to an increase (decrease) in its temperature. The mathematical expression for this idea, an energy balance, is as follows.

$$\text{rate of energy in} - \text{rate of energy out} = \text{rate of energy accumulation}$$

The flux is defined to be the rate that thermal energy flows through the rod per unit area, and we denote it by $\psi = \psi(x, t)$. If we let U represent the amount of energy in the rod, then the energy balance over it is

$$A(a)\psi(a, t) - A(b)\psi(b, t) = \left.\frac{dU}{dt}\right|_{\text{rod}}.$$

Factor a minus sign from the left side.

$$-[A(b)\psi(b, t) - A(a)\psi(a, t)] = \left.\frac{dU}{dt}\right|_{\text{rod}}$$

By the fundamental theorem of calculus, the term in square brackets is an integral.

$$-\int_a^b \frac{\partial}{\partial x}[A(x)\psi(x)]\,dx = \left.\frac{dU}{dt}\right|_{\text{rod}}$$

The thermal energy in the rod U is obtained by integrating the thermal energy density $e(x, t)$ over the rod's volume.

$$-\int_a^b \frac{\partial}{\partial x}[A(x)\psi(x)]\,dx = \frac{d}{dt}\int_{\text{rod}} e(x, t)\,dV$$

The volume differential is $dV = A(x)\,dx$.

$$-\int_a^b \frac{\partial}{\partial x}[A(x)\psi(x)]\,dx = \frac{d}{dt}\int_a^b e(x, t)A(x)\,dx$$

The thermal energy density is the mass density ρ times specific heat c times temperature $u(x, t)$.

$$-\int_a^b \frac{\partial}{\partial x}[A(x)\psi(x)]\,dx = \frac{d}{dt}\int_a^b \rho c u(x, t)A(x)\,dx$$

Bring the minus sign and derivative inside the integrals.

$$\int_a^b \left\{-\frac{\partial}{\partial x}[A(x)\psi(x)]\right\} dx = \int_a^b \rho c \frac{\partial u}{\partial t} A(x)\, dx$$

The integrands must be equal to one another.

$$-\frac{\partial}{\partial x}[A(x)\psi(x)] = \rho c \frac{\partial u}{\partial t} A(x)$$

According to Fourier's law of conduction, the heat flux is proportional to the temperature gradient.

$$\psi \propto \frac{\partial u}{\partial x}$$

In order to turn this into an equation we can use, we introduce a proportionality constant $-K_0$.

$$\psi = -K_0 \frac{\partial u}{\partial x}$$

K_0 is known as the thermal conductivity, and the minus sign is included because heat flows down a temperature gradient (that is, from a hot location to a cold location). As a result, the energy balance becomes an equation solely for the temperature.

$$-\frac{\partial}{\partial x}\left[-K_0 A(x) \frac{\partial u}{\partial x}\right] = \rho c \frac{\partial u}{\partial t} A(x)$$

Substitute $A(x) = b(1 - x/l)^2$ here.

$$\frac{\partial}{\partial x}\left[K_0 b \left(1 - \frac{x}{l}\right)^2 \frac{\partial u}{\partial x}\right] = \rho c b \left(1 - \frac{x}{l}\right)^2 \frac{\partial u}{\partial t}$$

Bring the constants in front of the derivative and divide both sides by $\rho c b$.

$$\frac{K_0}{\rho c} \frac{\partial}{\partial x}\left[\left(1 - \frac{x}{l}\right)^2 \frac{\partial u}{\partial x}\right] = \left(1 - \frac{x}{l}\right)^2 \frac{\partial u}{\partial t}$$

The governing equation for the temperature is then

$$\left(1 - \frac{x}{l}\right)^2 \frac{\partial u}{\partial t} = k \frac{\partial}{\partial x}\left[\left(1 - \frac{x}{l}\right)^2 \frac{\partial u}{\partial x}\right], \tag{1}$$

where $k = K_0/\rho c$ is another constant known as the thermal diffusivity. Multiply both sides by l^2 to eliminate the fractions in the equation.

$$(l - x)^2 \frac{\partial u}{\partial t} = k \frac{\partial}{\partial x}\left[(l - x)^2 \frac{\partial u}{\partial x}\right]$$

The initial temperature distribution in the rod is prescribed.

$$\text{Initial Condition:} \quad u(x, 0) = \phi(x)$$

The flat end at $x = 0$ is maintained at 0°.

$$\text{Boundary Condition 1:} \quad u(0, t) = 0$$

Because the rod is shaped like a cone and its sides are insulated, the pointy end at $x = l$ is effectively insulated as well.

$$\text{Boundary Condition 2:} \quad u_x(l, t) = 0$$

Since the PDE and its associated boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve it. Assume a product solution of the form $u = X(x)T(t)$ and substitute it into the PDE

$$(l-x)^2 \frac{\partial u}{\partial t} = k \frac{\partial}{\partial x}\left[(l-x)^2 \frac{\partial u}{\partial x}\right] \quad \rightarrow \quad (l-x)^2 X T' = k \frac{\partial}{\partial x}\left[(l-x)^2 X' T\right]$$

and the boundary conditions.

$$\begin{aligned} u(0,t) = 0 \quad &\rightarrow \quad X(0)T(t) = 0 \quad \rightarrow \quad X(0) = 0 \\ u_x(l,t) = 0 \quad &\rightarrow \quad X'(l)T(t) = 0 \quad \rightarrow \quad X'(l) = 0 \end{aligned}$$

In order to simplify the transformed PDE, we make the additional substitution $V(x) = (l-x)X(x)$.

$$V = (l-x)X \quad \rightarrow \quad \begin{cases} (l-x)V = (l-x)^2 X \\ X = \dfrac{V}{l-x} \end{cases}$$

$$V' = (l-x)X' - X \quad \rightarrow \quad V' + \frac{V}{l-x} = (l-x)X' \quad \rightarrow \quad (l-x)V' + V = (l-x)^2 X'$$

As a result, the PDE becomes

$$\begin{aligned} (l-x)VT' &= kT \frac{d}{dx}[(l-x)V' + V] \\ (l-x)VT' &= kT[-\cancel{V'} + (l-x)V'' + \cancel{V'}] \\ (l-x)VT' &= kT(l-x)V'' \\ VT' &= kTV''. \end{aligned}$$

Now separate variables in the PDE: bring the constants and functions of t to the left side and the functions of x to the right side. The final answer would be the same if k were put on the right side.

$$\frac{T'}{kT} = \frac{V''}{V}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant.

$$\frac{T'}{kT} = \frac{V''}{V} = \lambda \tag{2}$$

Values of λ for which the boundary conditions are satisfied are called the eigenvalues, and the nontrivial functions associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Suppose that λ is positive. Then equation (2) becomes

$$\frac{V''}{V} = \mu^2.$$

Multiply both sides by V.

$$V'' = \mu^2 V$$

The solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$V(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} V(0) &= C_1 = (l-0)X(0) = 0 \\ V(l) &= C_1 \cosh \mu l + C_2 \sinh \mu l = (l-l)X(l) = 0 \end{aligned}$$

Since $C_1 = 0$, the second equation reduces to $C_2 \sinh \mu l = 0$. Hyperbolic sine is not oscillatory, so the only way this equation is satisfied is if $C_2 = 0$. The trivial solution is obtained, so there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Suppose that λ is zero. Then equation (2) becomes

$$\frac{V''}{V} = 0.$$

Multiply both sides by V.

$$V'' = 0$$

The general solution is obtained by integrating both sides with respect to x twice.

$$V(x) = C_3 x + C_4$$

Apply the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} V(0) &= C_4 = (l-0)X(0) = 0 \\ V(l) &= C_3 l + C_4 = (l-l)X(l) = 0 \end{aligned}$$

Since $C_4 = 0$, the second equation reduces to $C_3 = 0$. The trivial solution is obtained, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Suppose that λ is negative. Then equation (2) becomes

$$\frac{V''}{V} = -\gamma^2.$$

Multiply both sides by V.

$$V'' = -\gamma^2 V$$

Its solution can be written in terms of sine and cosine.

$$V(x) = C_5 \cos \gamma x + C_6 \sin \gamma x$$

Apply the boundary conditions to determine C_5 and C_6.

$$V(0) = C_5 = (l-0)X(0) = 0$$
$$V(l) = C_5 \cos\gamma l + C_6 \sin\gamma l = (l-l)X(l) = 0$$

Since $C_5 = 0$, the second equation reduces to $C_6 \sin\gamma l = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Then

$$\sin\gamma l = 0$$
$$\gamma l = n\pi, \quad n = 1, 2, \ldots$$
$$\gamma_n = \frac{n\pi}{l}, \quad n = 1, 2, \ldots.$$

The eigenfunctions associated with these eigenvalues for λ are

$$V(x) = C_6 \sin\gamma x \quad \rightarrow \quad V_n(x) = \sin\frac{n\pi x}{l}, \quad n = 1, 2, \ldots.$$

Now the related ODE for T will be solved with $\lambda = -(n\pi/l)^2$.

$$\frac{T'}{kT} = -\frac{n^2\pi^2}{l^2}$$

Multiply both sides by kT.

$$T' = -k\frac{n^2\pi^2}{l^2}T$$

The general solution is written in terms of the exponential function.

$$T(t) = C_7 \exp\left(-k\frac{n^2\pi^2}{l^2}t\right)$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of the eigenfunctions over all the eigenvalues.

$$u(x,t) = \sum_{n=1}^{\infty} B_n X_n(x) T_n(t)$$
$$= \sum_{n=1}^{\infty} B_n \frac{V_n(x)}{l-x} T_n(t)$$

Therefore, the temperature in the conical rod is

$$\boxed{u(x,t) = \frac{1}{l-x}\sum_{n=1}^{\infty} B_n \exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\sin\frac{n\pi x}{l}.}$$

Now the initial condition will be used to determine the coefficients B_n in terms of $\phi(x)$.

$$u(x,0) = \frac{1}{l-x}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \phi(x)$$

To solve for B_n, multiply both sides by $(l-x)\sin(m\pi x/l)$, where m is an integer,

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l} = (l-x)\phi(x)\sin\frac{m\pi x}{l}$$

and then integrate both sides with respect to x from 0 to l.

$$\int_0^l \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\,dx = \int_0^l (l-x)\phi(x)\sin\frac{m\pi x}{l}\,dx$$

Bring the constants in front of the integral on the left side.

$$\sum_{n=1}^{\infty} B_n \int_0^l \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\,dx = \int_0^l (l-x)\phi(x)\sin\frac{m\pi x}{l}\,dx$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$B_n \int_0^l \sin^2\frac{n\pi x}{l}\,dx = \int_0^l (l-x)\phi(x)\sin\frac{n\pi x}{l}\,dx$$

Evaluate the integral on the left side.

$$B_n \cdot \frac{l}{2} = \int_0^l (l-x)\phi(x)\sin\frac{n\pi x}{l}\,dx$$

Therefore,

$$\boxed{B_n = \frac{2}{l}\int_0^l (l-x)\phi(x)\sin\frac{n\pi x}{l}\,dx.}$$