Exercise 11

Write out the solution of problem (11) explicitly, starting from the discussion in Section 5.6.

Solution

Problem (11) is stated on page 149.

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f(x,t) \\ u(0,t) = h(t) \qquad & u(l,t) = k(t) \\ u(x,0) = \phi(x) \qquad & u_t(x,0) = \psi(x) \end{aligned} \tag{11}$$

In order to make the boundary conditions homogeneous, start by making the change of variables $v(x,t) = u(x,t) - r(x,t)$, where $r(x,t)$ is some function that satisfies the boundary conditions.

$$\begin{aligned} r(0,t) &= h(t) \\ r(l,t) &= k(t) \end{aligned}$$

A suitable function is

$$r(x,t) = \frac{l-x}{l}h(t) + \frac{x}{l}k(t).$$

As a result, the initial and boundary conditions for v are

$$\begin{aligned} v(0,t) &= u(0,t) - r(0,t) = h(t) - h(t) = 0 \\ v(l,t) &= u(l,t) - r(l,t) = k(t) - k(t) = 0 \\ v(x,0) &= u(x,0) - r(x,0) = \phi(x) - \frac{l-x}{l}h(0) - \frac{x}{l}k(0) \\ v_t(x,0) &= u_t(x,0) - r_t(x,0) = \psi(x) - \frac{l-x}{l}h'(0) - \frac{x}{l}k'(0). \end{aligned}$$

Substitute $u(x,t) = v(x,t) + r(x,t)$ into the PDE now to find the one that v satisfies.

$$(v_{tt} + r_{tt}) - c^2(v_{xx} + r_{xx}) = f(x,t)$$

Distribute c^2 and bring r_{tt} and $c^2 r_{xx}$ to the right side.

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= f(x,t) - r_{tt} + \underbrace{c^2 r_{xx}}_{=\,0} \\ &= f(x,t) - \frac{l-x}{l}h''(t) - \frac{x}{l}k''(t) \end{aligned}$$

Consequently, the PDE for v is

$$v_{tt} - c^2 v_{xx} = Q(x,t), \tag{1}$$

where

$$Q(x,t) = f(x,t) - \frac{l-x}{l}h''(t) - \frac{x}{l}k''(t).$$

Since the PDE is linear and inhomogeneous, we choose to apply the method of eigenfunction expansion to solve it. Consider the eigenvalue problem of the differential operator involving the spatial variable x

$$\frac{d^2}{dx^2}\zeta = \lambda\zeta \tag{2}$$

with the same boundary conditions as v.

$$\zeta(0) = 0$$
$$\zeta(l) = 0$$

Values of λ for which the boundary conditions are satisfied are known as the eigenvalues, and the nontrivial solutions associated with them are called the eigenfunctions. Equation (2) is known as the one-dimensional Helmholtz equation; the eigenfunctions for it are known to be orthogonal and form a complete set, which will prove useful later.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Suppose that λ is positive. Then equation (2) becomes

$$\frac{d^2\zeta}{dx^2} = \mu^2\zeta.$$

Its solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\zeta(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Apply the boundary conditions to determine C_1 and C_2.

$$\zeta(0) = C_1 = 0$$
$$\zeta(l) = C_1 \cosh \mu l + C_2 \sinh \mu l = 0$$

Since $C_1 = 0$, the second equation reduces to $C_2 \sinh \mu l = 0$. Hyperbolic sine is not oscillatory, so the only way this equation is satisfied is if $C_2 = 0$. The trivial solution is obtained, so there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Suppose that λ is zero. Then equation (2) becomes

$$\frac{d^2\zeta}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\zeta(x) = C_3 x + C_4$$

Apply the boundary conditions to determine C_3 and C_4.

$$\zeta(0) = C_4 = 0$$
$$\zeta(l) = C_3 l + C_4 = 0$$

Since $C_4 = 0$, the second equation reduces to $C_3 = 0$. The trivial solution is obtained, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Suppose that λ is negative. Then equation (3) becomes

$$\frac{d^2\zeta}{dx^2} = -\gamma^2\zeta.$$

Its solution can be written in terms of sine and cosine.

$$\zeta(x) = C_5 \cos\gamma x + C_6 \sin\gamma x$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned}\zeta(0) &= C_5 = 0\\ \zeta(l) &= C_5 \cos\gamma l + C_6 \sin\gamma l = 0\end{aligned}$$

Since $C_5 = 0$, the second equation reduces to $C_6 \sin\gamma l = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned}\sin\gamma l &= 0\\ \gamma l &= n\pi, \quad n = 1, 2, \ldots\\ \gamma_n &= \frac{n\pi}{l}, \quad n = 1, 2, \ldots.\end{aligned}$$

The eigenfunctions associated with these eigenvalues for λ are

$$\zeta(x) = C_6 \sin\gamma x \quad \to \quad \zeta_n(x) = \sin\frac{n\pi x}{l}, \quad n = 1, 2, \ldots.$$

Method 1 - Using Term-by-Term Differentiation

The eigenfunctions of the Helmholtz equation form a complete set, so the unknown function v can be expanded in terms of them.

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi x}{l}$$

To determine the generalized Fourier coefficients $a_n(t)$, substitute this expansion into the PDE.

$$v_{tt} - c^2 v_{xx} = Q(x,t)$$

$$\frac{\partial^2}{\partial t^2}\sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi x}{l} - c^2 \frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi x}{l} = Q(x,t)$$

Because v satisfies homogeneous boundary conditions and v, $\partial v/\partial x$, and $\partial v/\partial t$ are continuous (reasonable assumptions for the displacement of a homogeneous elastic string), the two series can in fact be differentiated term by term.

$$\sum_{n=1}^{\infty} \frac{d^2 a_n}{dt^2} \sin\frac{n\pi x}{l} - c^2 \sum_{n=1}^{\infty} a_n(t) \frac{d^2}{dx^2} \sin\frac{n\pi x}{l} = Q(x,t)$$

The operator applied to the eigenfunction is equal to the eigenvalue times the eigenfunction.

$$\sum_{n=1}^{\infty} \frac{d^2 a_n}{dt^2} \sin\frac{n\pi x}{l} - c^2 \sum_{n=1}^{\infty} a_n(t)\lambda_n \sin\frac{n\pi x}{l} = Q(x,t)$$

Combine the series on the left side.

$$\sum_{n=1}^{\infty}\left[\frac{d^2a_n}{dt^2} - c^2\lambda_n a_n(t)\right]\sin\frac{n\pi x}{l} = Q(x,t)$$

The left side is essentially a Fourier sine series expansion of $Q(x,t)$. To solve for the term in square brackets, multiply both sides by $\sin(m\pi x/l)$, where m is an integer,

$$\sum_{n=1}^{\infty}\left[\frac{d^2a_n}{dt^2} - c^2\lambda_n a_n(t)\right]\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l} = Q(x,t)\sin\frac{m\pi x}{l}$$

and then integrate both sides with respect to x from 0 to l.

$$\int_0^l \sum_{n=1}^{\infty}\left[\frac{d^2a_n}{dt^2} - c^2\lambda_n a_n(t)\right]\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \int_0^l Q(x,t)\sin\frac{m\pi x}{l}\,dx$$

Bring the functions of t in front of the integral on the left side and substitute $Q(x,t)$ on the right side.

$$\sum_{n=1}^{\infty}\left[\frac{d^2a_n}{dt^2} - c^2\lambda_n a_n(t)\right]\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \int_0^l \left[f(x,t) - \frac{l-x}{l}h''(t) - \frac{x}{l}k''(t)\right]\sin\frac{m\pi x}{l}\,dx$$

Since the eigenfunctions are orthogonal, the integral on the left side is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$. Split up the integral on the right side and bring the functions of t in front of them.

$$\begin{aligned}\left[\frac{d^2a_n}{dt^2} - c^2\lambda_n a_n(t)\right]&\int_0^l \sin^2\frac{n\pi x}{l}\,dx \\ &= \int_0^l f(x,t)\sin\frac{n\pi x}{l}\,dx - \frac{h''(t)}{l}\int_0^l (l-x)\sin\frac{n\pi x}{l}\,dx - \frac{k''(t)}{l}\int_0^l x\sin\frac{n\pi x}{l}\,dx\end{aligned}$$

Evaluate the integrals that can be evaluated.

$$\begin{aligned}\left[\frac{d^2a_n}{dt^2} - c^2\lambda_n a_n(t)\right]\cdot\frac{l}{2} &= \int_0^l f(x,t)\sin\frac{n\pi x}{l}\,dx - \frac{h''(t)}{l}\left(\frac{l^2}{n\pi}\right) - \frac{k''(t)}{l}\left[-\frac{(-1)^n l^2}{n\pi}\right] \\ &= \int_0^l f(x,t)\sin\frac{n\pi x}{l}\,dx + \frac{l}{n\pi}[(-1)^n k''(t) - h''(t)]\end{aligned}$$

Multiply both sides by $2/l$ and replace λ_n with $-(n\pi/l)^2$.

$$\frac{d^2a_n}{dt^2} + c^2\frac{n^2\pi^2}{l^2}a_n = \frac{2}{l}\int_0^l f(x,t)\sin\frac{n\pi x}{l}\,dx + \frac{2}{n\pi}[(-1)^n k''(t) - h''(t)]$$

For the time being, let the right side be denoted by $R(t)$.

$$\frac{d^2a_n}{dt^2} + c^2\frac{n^2\pi^2}{l^2}a_n = R(t)$$

With the help of the method of eigenfunction expansion, the PDE has been reduced to a second-order inhomogeneous ODE. Because the ODE is linear, the general solution is the sum of a complementary solution and a particular solution.

$$a_n = a_c + a_p$$

The complementary solution satisfies the associated homogeneous equation.

$$\frac{d^2a_c}{dt^2} + c^2\frac{n^2\pi^2}{l^2}a_c = 0$$

Its general solution can be written in terms of sine and cosine.

$$a_c(t) = C_7 \cos\frac{cn\pi t}{l} + C_8 \sin\frac{cn\pi t}{l}$$

The method of variation of parameters will be applied to determine a particular solution. Allow the parameters in the complementary solution to vary,

$$a_p(t) = b_1(t)\cos\frac{cn\pi t}{l} + b_2(t)\sin\frac{cn\pi t}{l}$$

find da_p/dt and d^2a_p/dt^2,

$$\begin{aligned}
\frac{da_p}{dt} &= b_1'(t)\cos\frac{cn\pi t}{l} + b_2'(t)\sin\frac{cn\pi t}{l} + \frac{cn\pi}{l}\left[-b_1(t)\sin\frac{cn\pi t}{l} + b_2(t)\cos\frac{cn\pi t}{l}\right] \\
\frac{d^2a_p}{dt^2} &= b_1''(t)\cos\frac{cn\pi t}{l} + b_2''(t)\sin\frac{cn\pi t}{l} + \frac{cn\pi}{l}\left[-b_1'(t)\sin\frac{cn\pi t}{l} + b_2'(t)\cos\frac{cn\pi t}{l}\right] \\
&\quad + \frac{cn\pi}{l}\left[-b_1'(t)\sin\frac{cn\pi t}{l} + b_2'(t)\cos\frac{cn\pi t}{l}\right] + \frac{c^2n^2\pi^2}{l^2}\left[-b_1(t)\cos\frac{cn\pi t}{l} - b_2(t)\sin\frac{cn\pi t}{l}\right] \\
&= b_1''(t)\cos\frac{cn\pi t}{l} + b_2''(t)\sin\frac{cn\pi t}{l} + \frac{2cn\pi}{l}\left[-b_1'(t)\sin\frac{cn\pi t}{l} + b_2'(t)\cos\frac{cn\pi t}{l}\right] \\
&\qquad - \frac{c^2n^2\pi^2}{l^2}\left[b_1(t)\cos\frac{cn\pi t}{l} + b_2(t)\sin\frac{cn\pi t}{l}\right]
\end{aligned}$$

and substitute these formulas into the inhomogeneous ODE that $a_p(t)$ satisfies.

$$\frac{d^2a_p}{dt^2} + c^2\frac{n^2\pi^2}{l^2}a_p = R(t)$$

$$\begin{aligned}
&b_1''(t)\cos\frac{cn\pi t}{l} + b_2''(t)\sin\frac{cn\pi t}{l} + \frac{2cn\pi}{l}\left[-b_1'(t)\sin\frac{cn\pi t}{l} + b_2'(t)\cos\frac{cn\pi t}{l}\right] \\
&\quad \cancel{- \frac{c^2n^2\pi^2}{l^2}\left[b_1(t)\cos\frac{cn\pi t}{l} + b_2(t)\sin\frac{cn\pi t}{l}\right]} + \cancel{c^2\frac{n^2\pi^2}{l^2}\left[b_1(t)\cos\frac{cn\pi t}{l} + b_2(t)\sin\frac{cn\pi t}{l}\right]} = R(t)
\end{aligned}$$

What remains is

$$b_1''(t)\cos\frac{cn\pi t}{l} + b_2''(t)\sin\frac{cn\pi t}{l} + \frac{2cn\pi}{l}\left[-b_1'(t)\sin\frac{cn\pi t}{l} + b_2'(t)\cos\frac{cn\pi t}{l}\right] = R(t).$$

If we set the term in square brackets equal to zero

$$-b_1'(t)\sin\frac{cn\pi t}{l} + b_2'(t)\cos\frac{cn\pi t}{l} = 0, \tag{3}$$

then the previous equation reduces to

$$b_1''(t)\cos\frac{cn\pi t}{l} + b_2''(t)\sin\frac{cn\pi t}{l} = R(t). \tag{4}$$

This is a system of two equations for two unknowns, $b_1(t)$ and $b_2(t)$. Solve equation (3) for $b_2'(t)$

$$b_2'(t) = b_1'(t)\frac{\sin\frac{cn\pi t}{l}}{\cos\frac{cn\pi t}{l}} \tag{5}$$

and substitute this formula into equation (4) to get an equation exclusively for b_1.

$$b_1''(t)\cos\frac{cn\pi t}{l} + \left[b_1'(t)\frac{\sin\frac{cn\pi t}{l}}{\cos\frac{cn\pi t}{l}}\right]'\sin\frac{cn\pi t}{l} = R(t)$$

Evaluate the derivative.

$$b_1''(t)\cos\frac{cn\pi t}{l} + b_1''(t)\frac{\sin\frac{cn\pi t}{l}}{\cos\frac{cn\pi t}{l}}\sin\frac{cn\pi t}{l} + b_1'(t)\frac{1}{\cos^2\frac{cn\pi t}{l}}\sin\frac{cn\pi t}{l} = R(t)$$

Factor the left side.

$$\frac{b_1''(t)}{\cos\frac{cn\pi t}{l}}\left(\cos^2\frac{cn\pi t}{l} + \sin^2\frac{cn\pi t}{l}\right) + b_1'(t)\frac{1}{\cos\frac{cn\pi t}{l}}\frac{\sin\frac{cn\pi t}{l}}{\cos\frac{cn\pi t}{l}} = R(t)$$

Replace the trigonometric functions.

$$b_1''(t)\sec\frac{cn\pi t}{l} + b_1'(t)\sec\frac{cn\pi t}{l}\tan\frac{cn\pi t}{l} = R(t)$$

The left side can be written as $d/dt[b_1'(t)\sec(cn\pi t/l)]$ by the product rule.

$$\frac{d}{dt}\left[b_1'(t)\sec\frac{cn\pi t}{l}\right] = R(t)$$

Integrate both sides from 0 to t and change secant back to cosine.

$$\frac{b_1'(t)}{\cos\frac{cn\pi t}{l}} = \int_0^t R(s)\,ds$$

Multiply both sides by $\cos(cn\pi t/l)$.

$$b_1'(t) = \cos\frac{cn\pi t}{l}\int_0^t R(s)\,ds \tag{6}$$

Integrate both sides from 0 to t once more.

$$\begin{aligned}b_1(t) &= \int_0^t \cos\frac{cn\pi q}{l}\int_0^q R(s)\,ds\,dq \\ &= \int_0^t\int_0^q \cos\frac{cn\pi q}{l}R(s)\,ds\,dq\end{aligned}$$

Since cosine can be integrated, the order of integration will be switched so that dq comes first.

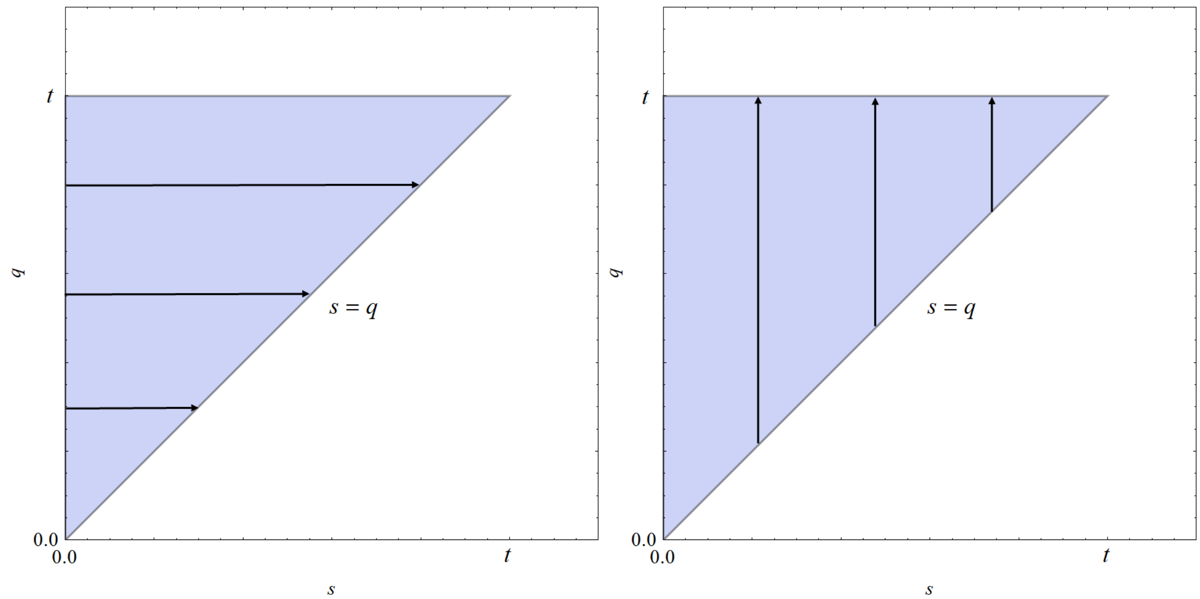

Figure 1: The current mode of integration in the sq-plane is shown on the left. The domain will be integrated over as shown on the right to switch the order of integration.

$$\begin{aligned}
b_1(t) &= \int_0^t \int_s^t \cos\frac{cn\pi q}{l} R(s)\,dq\,ds \\
&= \int_0^t \frac{l}{cn\pi} \sin\frac{cn\pi q}{l}\bigg|_s^t R(s)\,ds \\
&= \frac{l}{cn\pi} \int_0^t \left(\sin\frac{cn\pi t}{l} - \sin\frac{cn\pi s}{l}\right) R(s)\,ds
\end{aligned}$$

Now combine equations (5) and (6) to get an equation for $b_2'(t)$.

$$b_2'(t) = \sin\frac{cn\pi t}{l} \int_0^t R(s)\,ds$$

Integrate both sides from 0 to t.

$$\begin{aligned}
b_2(t) &= \int_0^t \sin\frac{cn\pi q}{l} \int_0^q R(s)\,ds\,dq \\
&= \int_0^t \int_0^q \sin\frac{cn\pi q}{l} R(s)\,ds\,dq \\
&= \int_0^t \int_s^t \sin\frac{cn\pi q}{l} R(s)\,dq\,ds \\
&= \int_0^t \left(-\frac{l}{cn\pi}\right) \cos\frac{cn\pi q}{l}\bigg|_s^t R(s)\,ds \\
&= \frac{l}{cn\pi} \int_0^t \left(\cos\frac{cn\pi s}{l} - \cos\frac{cn\pi t}{l}\right) R(s)\,ds
\end{aligned}$$

Consequently, the particular solution is

$$\begin{aligned}
a_p(t) &= b_1(t)\cos\frac{cn\pi t}{l} + b_2(t)\sin\frac{cn\pi t}{l} \\
&= \frac{l}{cn\pi}\cos\frac{cn\pi t}{l}\int_0^t\left(\sin\frac{cn\pi t}{l} - \sin\frac{cn\pi s}{l}\right)R(s)\,ds + \frac{l}{cn\pi}\sin\frac{cn\pi t}{l}\int_0^t\left(\cos\frac{cn\pi s}{l} - \cos\frac{cn\pi t}{l}\right)R(s)\,ds \\
&= \frac{l}{cn\pi}\int_0^t\left(\cancel{\sin\frac{cn\pi t}{l}\cos\frac{cn\pi t}{l}} - \sin\frac{cn\pi s}{l}\cos\frac{cn\pi t}{l} + \sin\frac{cn\pi t}{l}\cos\frac{cn\pi s}{l} - \cancel{\sin\frac{cn\pi t}{l}\cos\frac{cn\pi t}{l}}\right)R(s)\,ds \\
&= \frac{l}{cn\pi}\int_0^t\sin\left[\frac{cn\pi}{l}(t-s)\right]R(s)\,ds.
\end{aligned}$$

Hence, the general solution for a_n is

$$a_n(t) = C_7\cos\frac{cn\pi t}{l} + C_8\sin\frac{cn\pi t}{l} + \frac{l}{cn\pi}\int_0^t\sin\left[\frac{cn\pi}{l}(t-s)\right]R(s)\,ds.$$

Use the initial conditions for v in combination with the eigenfunction expansion to determine those for a_n.

$$v(x,0) = \sum_{n=1}^{\infty} a_n(0)\sin\frac{n\pi x}{l} = \phi(x) - \frac{l-x}{l}h(0) - \frac{x}{l}k(0) \tag{7}$$

$$v_t(x,0) = \sum_{n=1}^{\infty} \frac{da_n}{dt}(0)\sin\frac{n\pi x}{l} = \psi(x) - \frac{l-x}{l}h'(0) - \frac{x}{l}k'(0) \tag{8}$$

To solve for $a_n(0)$, multiply both sides of equation (7) by $\sin(m\pi x/l)$

$$\sum_{n=1}^{\infty} a_n(0)\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l} = \left[\phi(x) - \frac{l-x}{l}h(0) - \frac{x}{l}k(0)\right]\sin\frac{m\pi x}{l}$$

and then integrate both sides with respect to x from 0 to l.

$$\int_0^l \sum_{n=1}^{\infty} a_n(0)\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \int_0^l \left[\phi(x) - \frac{l-x}{l}h(0) - \frac{x}{l}k(0)\right]\sin\frac{m\pi x}{l}\,dx$$

Bring the constants in front of the integral on the left side and split up the integral on the right side.

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n(0)\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx \\
&\qquad = \int_0^l \phi(x)\sin\frac{m\pi x}{l}\,dx - \frac{h(0)}{l}\int_0^l (l-x)\sin\frac{m\pi x}{l}\,dx - \frac{k(0)}{l}\int_0^l x\sin\frac{m\pi x}{l}\,dx
\end{aligned}$$

Since the eigenfunctions are orthogonal, the integral on the left side is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$a_n(0)\int_0^l \sin^2\frac{n\pi x}{l}\,dx = \int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx - \frac{h(0)}{l}\int_0^l (l-x)\sin\frac{n\pi x}{l}\,dx - \frac{k(0)}{l}\int_0^l x\sin\frac{n\pi x}{l}\,dx$$

Evaluate the integrals that can be evaluated.

$$a_n(0)\cdot\frac{l}{2} = \int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx - \frac{h(0)}{l}\left(\frac{l^2}{n\pi}\right) - \frac{k(0)}{l}\left[-\frac{(-1)^n l^2}{n\pi}\right]$$

The first initial condition for a_n is then

$$a_n(0) = \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx + \frac{2}{n\pi}[(-1)^n k(0) - h(0)].$$

To obtain the second one, multiply both sides of equation (8) by $\sin(m\pi x/l)$

$$\sum_{n=1}^{\infty}\frac{da_n}{dt}(0)\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l} = \left[\psi(x) - \frac{l-x}{l}h'(0) - \frac{x}{l}k'(0)\right]\sin\frac{m\pi x}{l}$$

and then integrate both sides with respect to x from 0 to l.

$$\int_0^l \sum_{n=1}^{\infty}\frac{da_n}{dt}(0)\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \int_0^l \left[\psi(x) - \frac{l-x}{l}h'(0) - \frac{x}{l}k'(0)\right]\sin\frac{m\pi x}{l}\,dx$$

Bring the constants in front of the integral on the left side and split up the integral on the right side.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{da_n}{dt}(0)\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx \\ = \int_0^l \psi(x)\sin\frac{m\pi x}{l}\,dx - \frac{h'(0)}{l}\int_0^l (l-x)\sin\frac{m\pi x}{l}\,dx - \frac{k'(0)}{l}\int_0^l x\sin\frac{m\pi x}{l}\,dx\end{aligned}$$

Since the eigenfunctions are orthogonal, the integral on the left side is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$\frac{da_n}{dt}(0)\int_0^l \sin^2\frac{n\pi x}{l}\,dx = \int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx - \frac{h'(0)}{l}\int_0^l (l-x)\sin\frac{n\pi x}{l}\,dx - \frac{k'(0)}{l}\int_0^l x\sin\frac{n\pi x}{l}\,dx$$

Evaluate the integrals that can be evaluated.

$$\frac{da_n}{dt}(0)\cdot\frac{l}{2} = \int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx - \frac{h'(0)}{l}\left(\frac{l^2}{n\pi}\right) - \frac{k'(0)}{l}\left[-\frac{(-1)^n l^2}{n\pi}\right]$$

The second initial condition for a_n is then

$$\frac{da_n}{dt}(0) = \frac{2}{l}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx + \frac{2}{n\pi}[(-1)^n k'(0) - h'(0)].$$

Now apply the two initial conditions to obtain a system of equations for C_7 and C_8.

$$\begin{aligned}a(0) = C_7 &= \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx + \frac{2}{n\pi}[(-1)^n k(0) - h(0)] \\ \frac{da_n}{dt}(0) = \frac{cn\pi}{l}(C_8) &= \frac{2}{l}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx + \frac{2}{n\pi}[(-1)^n k'(0) - h'(0)]\end{aligned}$$

Solving the second equation for C_8 gives

$$C_8 = \frac{2}{cn\pi}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx + \frac{2l}{cn^2\pi^2}[(-1)^n k'(0) - h'(0)].$$

Plugging in the formulas for C_7, C_8, and $R(t)$ and simplifying, the general solution for $a_n(t)$ becomes

$$\begin{aligned} a_n(t) &= \frac{2}{n\pi}\left\{[(-1)^n k(0) - h(0)]\cos\frac{cn\pi t}{l} + \frac{l}{cn\pi}[(-1)^n k'(0) - h'(0)]\sin\frac{cn\pi t}{l}\right\} \\ &\quad + \frac{2}{l}\int_0^l \left[\phi(q)\sin\frac{n\pi q}{l}\cos\frac{cn\pi t}{l} + \frac{l}{cn\pi}\psi(q)\sin\frac{n\pi q}{l}\sin\frac{cn\pi t}{l}\right] dq \\ &\qquad + \frac{l}{cn\pi}\int_0^t \sin\left[\frac{cn\pi}{l}(t-s)\right]\left\{\frac{2}{l}\int_0^l f(q,s)\sin\frac{n\pi q}{l}\,dq + \frac{2}{n\pi}[(-1)^n k''(s) - h''(s)]\right\} ds. \end{aligned}$$

Therefore, since $u(x,t) = r(x,t) + v(x,t)$,

$$\boxed{u(x,t) = \frac{l-x}{l}h(t) + \frac{x}{l}k(t) + \sum_{n=1}^{\infty} a_n(t)\sin\frac{n\pi x}{l}, \quad 0 \le x \le l.}$$

Method 2 - Without Using Term-by-Term Differentiation

The eigenfunctions of the Helmholtz equation are known to form a complete set, so all of the functions in the PDE for v can be expanded in terms of them.

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty} A_n(t)\zeta_n(x) &\to\quad v\zeta_m &= \sum_{n=1}^{\infty} A_n\zeta_n\zeta_m &\to\quad \int_0^l v\zeta_n\,dx &= A_n\int_0^l \zeta_n^2\,dx = A_n\cdot\frac{l}{2} \\ \frac{\partial^2 v}{\partial t^2} &= \sum_{n=1}^{\infty} B_n(t)\zeta_n(x) &\to\quad \frac{\partial^2 v}{\partial t^2}\zeta_m &= \sum_{n=1}^{\infty} B_n\zeta_n\zeta_m &\to\quad \int_0^l \frac{\partial^2 v}{\partial t^2}\zeta_n\,dx &= B_n\int_0^l \zeta_n^2\,dx = B_n\cdot\frac{l}{2} \\ Q(x,t) &= \sum_{n=1}^{\infty} D_n(t)\zeta_n(x) &\to\quad Q(x,t)\zeta_m &= \sum_{n=1}^{\infty} D_n\zeta_n\zeta_m &\to\quad \int_0^l Q(x,t)\zeta_n\,dx &= D_n\int_0^l \zeta_n^2\,dx = D_n\cdot\frac{l}{2} \\ \frac{\partial^2 v}{\partial x^2} &= \sum_{n=1}^{\infty} E_n(t)\zeta_n(x) &\to\quad \frac{\partial^2 v}{\partial x^2}\zeta_m &= \sum_{n=1}^{\infty} E_n\zeta_n\zeta_m &\to\quad \int_0^l \frac{\partial^2 v}{\partial x^2}\zeta_n\,dx &= E_n\int_0^l \zeta_n^2\,dx = E_n\cdot\frac{l}{2} \end{aligned}$$

It should be emphasized that these are generalized Fourier series expansions for the functions, not product solutions that come about from using the method of separation of variables. Solve the latter equations for the generalized Fourier coefficients.

$$\begin{aligned} A_n(t) &= \frac{2}{l}\int_0^l v\zeta_n\,dx \\ B_n(t) &= \frac{2}{l}\int_0^l \frac{\partial^2 v}{\partial t^2}\zeta_n\,dx = \frac{d^2}{dt^2}\left(\frac{2}{l}\int_0^l v\zeta_n\,dx\right) = \frac{d^2 A_n}{dt^2} \\ D_n(t) &= \frac{2}{l}\int_0^l Q(x,t)\zeta_n\,dx \\ E_n(t) &= \frac{2}{l}\int_0^l \frac{\partial^2 v}{\partial x^2}\zeta_n\,dx = \frac{2}{l}\left(\underbrace{\frac{\partial v}{\partial x}\zeta_n\bigg|_0^l}_{=\,0} - \int_0^l \frac{\partial v}{\partial x}\frac{d\zeta_n}{dx}\,dx\right) = -\frac{2n\pi}{l^2}\int_0^l \frac{\partial v}{\partial x}\cos\frac{n\pi x}{l}\,dx \end{aligned}$$

Apply integration by parts once more in order to write E_n in terms of A_n.

$$\begin{aligned}
E_n(t) &= -\frac{2n\pi}{l^2}\Bigg[\underbrace{v\cos\frac{n\pi x}{l}\bigg|_0^l}_{=0} - \int_0^l v\left(-\frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)dx\Bigg] \\
&= -\frac{n^2\pi^2}{l^2}\left(\frac{2}{l}\int_0^l v\sin\frac{n\pi x}{l}\,dx\right) \\
&= -\frac{n^2\pi^2}{l^2}A_n
\end{aligned}$$

Now that the coefficients are known, substitute the eigenfunction expansions into the PDE.

$$\begin{aligned}
v_{tt} - c^2 v_{xx} &= Q(x,t) \\
\sum_{n=1}^{\infty} B_n(t)\zeta_n(x) - c^2\sum_{n=1}^{\infty} E_n(t)\zeta_n(x) &= \sum_{n=1}^{\infty} D_n(t)\zeta_n(x) \\
\sum_{n=1}^{\infty}[B_n(t) - c^2E_n(t)]\zeta_n(x) &= \sum_{n=1}^{\infty} D_n(t)\zeta_n(x)
\end{aligned}$$

Thus,

$$B_n(t) - c^2E_n(t) = D_n(t).$$

Substitute the formulas for B_n, E_n, and D_n to obtain an ODE for A_n exclusively.

$$\frac{d^2A_n}{dt^2} + c^2\frac{n^2\pi^2}{l^2}A_n = \frac{2}{l}\int_0^l Q(x,t)\zeta_n\,dx$$

Substitute $Q(x,t)$ and ζ_n and evaluate the resulting integral.

$$\begin{aligned}
\frac{d^2A_n}{dt^2} + c^2\frac{n^2\pi^2}{l^2}A_n &= \frac{2}{l}\int_0^l\left\{f(x,t) - \frac{l-x}{l}h''(t) - \frac{x}{l}k''(t)\right\}\sin\frac{n\pi x}{l}\,dx \\
&= \frac{2}{l}\int_0^l f(x,t)\sin\frac{n\pi x}{l}\,dx - \frac{2h''(t)}{l^2}\int_0^l (l-x)\sin\frac{n\pi x}{l}\,dx - \frac{2k''(t)}{l^2}\int_0^l x\sin\frac{n\pi x}{l}\,dx \\
&= \frac{2}{l}\int_0^l f(x,t)\sin\frac{n\pi x}{l}\,dx - \frac{2h''(t)}{l^2}\left(\frac{l^2}{n\pi}\right) - \frac{2k''(t)}{l^2}\left[-\frac{(-1)^n l^2}{n\pi}\right] \\
&= \frac{2}{l}\int_0^l f(x,t)\sin\frac{n\pi x}{l}\,dx + \frac{2}{n\pi}[(-1)^n k''(t) - h''(t)] \\
&= R(t)
\end{aligned}$$

This is the same ODE that was obtained for a_n in Method 1. The initial conditions are also the same as before, so $A_n(t) = a_n(t)$ and the same solution is obtained for v and u.

Method 3 - Mr. Strauss's Way

Here the method of eigenfunction expansion will be applied directly to the inhomogeneous PDE for u. The eigenfunctions of the Helmholtz equation are known to form a complete set, so all the functions in the PDE can be expanded in terms of them.

$$u(x,t) = \sum_{n=1}^{\infty} H_n(t)\zeta_n(x) \quad \to \quad u\zeta_m = \sum_{n=1}^{\infty} H_n\zeta_n\zeta_m \quad \to \quad \int_0^l u\zeta_n\,dx = H_n\int_0^l \zeta_n^2\,dx = H_n\cdot\frac{l}{2}$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} J_n(t)\zeta_n(x) \quad \to \quad \frac{\partial u}{\partial t}\zeta_m = \sum_{n=1}^{\infty} J_n\zeta_n\zeta_m \quad \to \quad \int_0^l \frac{\partial u}{\partial t}\zeta_n\,dx = J_n\int_0^l \zeta_n^2\,dx = J_n\cdot\frac{l}{2}$$

$$\frac{\partial^2 u}{\partial t^2} = \sum_{n=1}^{\infty} e_n(t)\zeta_n(x) \quad \to \quad \frac{\partial^2 u}{\partial t^2}\zeta_m = \sum_{n=1}^{\infty} e_n\zeta_n\zeta_m \quad \to \quad \int_0^l \frac{\partial^2 u}{\partial t^2}\zeta_n\,dx = e_n\int_0^l \zeta_n^2\,dx = e_n\cdot\frac{l}{2}$$

$$f(x,t) = \sum_{n=1}^{\infty} F_n(t)\zeta_n(x) \quad \to \quad f(x,t)\zeta_m = \sum_{n=1}^{\infty} F_n\zeta_n\zeta_m \quad \to \quad \int_0^l f(x,t)\zeta_n\,dx = F_n\int_0^l \zeta_n^2\,dx = F_n\cdot\frac{l}{2}$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} G_n(t)\zeta_n(x) \quad \to \quad \frac{\partial^2 u}{\partial x^2}\zeta_m = \sum_{n=1}^{\infty} G_n\zeta_n\zeta_m \quad \to \quad \int_0^l \frac{\partial^2 u}{\partial x^2}\zeta_n\,dx = G_n\int_0^l \zeta_n^2\,dx = G_n\cdot\frac{l}{2}$$

It should be emphasized that these are generalized Fourier series expansions for the functions, not product solutions that come about from using the method of separation of variables. Solve the latter equations for the generalized Fourier coefficients.

$$\begin{aligned}
H_n(t) &= \frac{2}{l}\int_0^l u\zeta_n\,dx \\
J_n(t) &= \frac{2}{l}\int_0^l \frac{\partial u}{\partial t}\zeta_n\,dx = \frac{d}{dt}\left(\frac{2}{l}\int_0^l u\zeta_n\,dx\right) = \frac{dH_n}{dt} \\
e_n(t) &= \frac{2}{l}\int_0^l \frac{\partial^2 u}{\partial t^2}\zeta_n\,dx = \frac{d^2}{dt^2}\left(\frac{2}{l}\int_0^l u\zeta_n\,dx\right) = \frac{d^2H_n}{dt^2} \\
F_n(t) &= \frac{2}{l}\int_0^l f(x,t)\zeta_n\,dx \\
G_n(t) &= \frac{2}{l}\int_0^l \frac{\partial^2 u}{\partial x^2}\zeta_n\,dx = \frac{2}{l}\Bigg(\underbrace{\frac{\partial u}{\partial x}\zeta_n\bigg|_0^l}_{=\,0} - \int_0^l \frac{\partial u}{\partial x}\frac{d\zeta_n}{dx}\,dx\Bigg) = -\frac{2n\pi}{l^2}\int_0^l \frac{\partial u}{\partial x}\cos\frac{n\pi x}{l}\,dx
\end{aligned}$$

Apply integration by parts once more in order to write G_n in terms of H_n.

$$\begin{aligned}
G_n(t) &= -\frac{2n\pi}{l^2}\left[u\cos\frac{n\pi x}{l}\bigg|_0^l - \int_0^l u\left(-\frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)dx\right] \\
&= -\frac{2n\pi}{l^2}\left[u(l,t)\cos n\pi - u(0,t) + \frac{n\pi}{l}\int_0^l u\sin\frac{n\pi x}{l}\,dx\right] \\
&= -\frac{2n\pi}{l^2}\left[(-1)^n k(t) - h(t) + \frac{n\pi}{2}\cdot\frac{2}{l}\int_0^l u\sin\frac{n\pi x}{l}\,dx\right] \\
&= -\frac{2n\pi}{l^2}\left[(-1)^n k(t) - h(t) + \frac{n\pi}{2}H_n\right]
\end{aligned}$$

Now that the coefficients are known, substitute the eigenfunction expansions into the PDE.

$$u_{tt} - c^2 u_{xx} = f(x,t)$$

$$\sum_{n=1}^{\infty} e_n(t)\zeta_n(x) - c^2 \sum_{n=1}^{\infty} G_n(t)\zeta_n(x) = \sum_{n=1}^{\infty} F_n(t)\zeta_n(x)$$

$$\sum_{n=1}^{\infty} [e_n(t) - c^2 G_n(t)]\zeta_n(x) = \sum_{n=1}^{\infty} F_n(t)\zeta_n(x)$$

Thus,

$$e_n(t) - c^2 G_n(t) = F_n(t).$$

Substitute the formulas for e_n, G_n, and F_n to obtain an ODE for H_n exclusively.

$$\frac{d^2 H_n}{dt^2} + c^2 \frac{2n\pi}{l^2}\left[(-1)^n k(t) - h(t) + \frac{n\pi}{2} H_n\right] = \frac{2}{l}\int_0^l f(x,t)\zeta_n\,dx$$

Replace ζ_n with $\sin(n\pi x/l)$ and bring the terms without H_n to the right side.

$$\frac{d^2 H_n}{dt^2} + c^2 \frac{n^2\pi^2}{l^2} H_n = \frac{2}{l}\int_0^l f(x,t)\sin\frac{n\pi x}{l}\,dx - c^2 \frac{2n\pi}{l^2}[(-1)^n k(t) - h(t)]$$

For the time being, let the right side be denoted as $S(t)$.

$$\frac{d^2 H_n}{dt^2} + c^2 \frac{n^2\pi^2}{l^2} H_n = S(t)$$

This is essentially the same ODE that a_n satisfies—only the inhomogeneous term is different, so the same general solution can be used here.

$$H_n(t) = C_9 \cos\frac{cn\pi t}{l} + C_{10}\sin\frac{cn\pi t}{l} + \frac{l}{cn\pi}\int_0^t \sin\left[\frac{cn\pi}{l}(t-s)\right] S(s)\,ds.$$

Use the initial conditions for u in combination with the eigenfunction expansion to determine those for H_n.

$$u(x,0) = \sum_{n=1}^{\infty} H_n(0)\zeta_n(x) = \phi(x) \quad \rightarrow \quad \phi(x)\zeta_m = \sum_{n=1}^{\infty} H_n(0)\zeta_n\zeta_m \quad \rightarrow \quad \int_0^l \phi(x)\zeta_n\,dx = H_n(0)\int_0^l \zeta_n^2\,dx = H_n(0)\cdot\frac{l}{2}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} J_n(0)\zeta_n(x) = \psi(x) \quad \rightarrow \quad \psi(x)\zeta_m = \sum_{n=1}^{\infty} \frac{dH_n}{dt}(0)\zeta_n\zeta_m \quad \rightarrow \quad \int_0^l \psi(x)\zeta_n\,dx = \frac{dH_n}{dt}(0)\int_0^l \zeta_n^2\,dx = \frac{dH_n}{dt}(0)\cdot\frac{l}{2}$$

Solve the latter equations for the initial conditions.

$$H_n(0) = \frac{2}{l}\int_0^l \phi(x)\zeta_n\,dx$$

$$\frac{dH_n}{dt}(0) = \frac{2}{l}\int_0^l \psi(x)\zeta_n\,dx$$

Now apply them to obtain a system of equations for C_9 and C_{10}.

$$H_n(0) = C_9 = \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx$$

$$\frac{dH_n}{dt}(0) = \frac{cn\pi}{l}(C_{10}) = \frac{2}{l}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx \quad \rightarrow \quad C_{10} = \frac{2}{cn\pi}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx$$

Plugging in the formulas for C_9, C_{10}, and $S(t)$ and simplifying, the general solution for $H_n(t)$ becomes

$$\begin{aligned} H_n(t) = \frac{2}{l}\int_0^l \left[\phi(q)\sin\frac{n\pi q}{l}\cos\frac{cn\pi t}{l} + \frac{l}{cn\pi}\psi(q)\sin\frac{n\pi q}{l}\sin\frac{cn\pi t}{l}\right] dq \\ + \frac{l}{cn\pi}\int_0^t \sin\left[\frac{cn\pi}{l}(t-s)\right]\left\{\frac{2}{l}\int_0^l f(q,s)\sin\frac{n\pi q}{l}\,dq - 2n\pi\frac{c^2}{l^2}[(-1)^n k(s) - h(s)]\right\} ds. \end{aligned}$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} H_n(t)\sin\frac{n\pi x}{l}, \quad 0 < x < l.$$

This solution does not satisfy the PDE at $x = 0$ and $x = l$ because $u(0,t) \neq \zeta(0) = 0$ and $u(l,t) \neq \zeta(l) = 0$. Also, it converges more slowly than the solution obtained using Method 1 and Method 2.