Exercise 10

Prove the uniqueness of the Dirichlet problem $\Delta u = f$ in D, $u = g$ on bdy D by the energy method. That is, after subtracting two solutions $w = u - v$, multiply the Laplace equation for w by w itself and use the divergence theorem.

Solution

Suppose that there are two functions, u and v, that satisfy the Dirichlet problem. Then we have

$$\Delta u = f, \quad u = g \text{ on bdy } D$$
$$\Delta v = f, \quad v = g \text{ on bdy } D.$$

Subtract both sides of the second PDE from those of the first.

$$\Delta u - \Delta v = f - f$$

Factor the Laplacian operator.

$$\Delta(u - v) = 0$$

Subtract both sides of the second boundary condition from those of the first.

$$u - v = g - g \quad \text{on bdy } D$$

$$u - v = 0 \quad \text{on bdy } D$$

We find that the difference of the two solutions satisfies the Laplace equation, and the boundary condition associated with it is homogeneous. Let w represent this difference: $w = u - v$.

$$\Delta w = 0, \quad w = 0 \text{ on bdy } D$$

Multiply both sides of the Laplace equation by w.

$$w\Delta w = 0$$

Integrate both sides over the volume of D.

$$\iiint_D w\Delta w \, dV = 0$$

Apply Green's first identity here (page 180).

$$\iint_{\text{bdy } D} w\frac{\partial w}{\partial n} \, dS - \iiint_D \nabla w \cdot \nabla w \, dV = 0$$

Since $w = 0$ on the boundary of D, the first integral on the left is zero. Note also that $\nabla w \cdot \nabla w = |\nabla w|^2$.

$$\iiint_D |\nabla w|^2 \, dV = 0$$

For this equation to be satisfied, it must be the case that $\nabla w = \mathbf{0}$, which means w is a constant in D. w is known to be zero on the boundary, so $w = 0$ throughout D. This implies that the two solutions to the Dirichlet problem are one and the same, $u = v$. Therefore, the solution is unique.