Exercise 2

Find the solutions that depend only on r of the equation $u_{xx} + u_{yy} + u_{zz} = k^2 u$, where k is a positive constant. (*Hint:* Substitute $u = v/r$.)

Solution

This PDE is known as the Helmholtz equation.

$$\nabla^2 u = k^2 u$$

Since we're looking for solutions that depend only on r in three dimensions, we choose to write the Laplacian operator in spherical coordinates (θ here represents the angle from the polar axis).

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2 u}{\partial \theta^2} + (\cot\theta)\frac{\partial u}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\right] = k^2 u$$

A spherically symmetric solution is one that only depends on r, $u = u(r)$. With this assumption the PDE simplifies to an ODE that can be solved relatively easily.

$$\frac{d^2 u}{dr^2} + \frac{2}{r}\frac{du}{dr} = k^2 u \tag{1}$$

Before making the substitution $u = v/r$ in equation (1), find du/dr and d^2u/dr^2 in terms of the new variable.

$$\begin{aligned}\frac{du}{dr} &= -\frac{v}{r^2} + \frac{1}{r}\frac{dv}{dr} \\ \frac{d^2u}{dr^2} &= 2\frac{v}{r^3} - \frac{1}{r^2}\frac{dv}{dr} - \frac{1}{r^2}\frac{dv}{dr} + \frac{1}{r}\frac{d^2v}{dr^2} \\ &= 2\frac{v}{r^3} - \frac{2}{r^2}\frac{dv}{dr} + \frac{1}{r}\frac{d^2v}{dr^2}\end{aligned}$$

Now make the substitution in equation (1). As a result, equation (1) becomes

$$2\frac{v}{r^3} - \frac{2}{r^2}\frac{dv}{dr} + \frac{1}{r}\frac{d^2v}{dr^2} + \frac{2}{r}\left(-\frac{v}{r^2} + \frac{1}{r}\frac{dv}{dr}\right) = k^2\frac{v}{r}.$$

Expand the left side and cancel equal terms.

$$\cancel{2\frac{v}{r^3}} - \cancel{\frac{2}{r^2}\frac{dv}{dr}} + \frac{1}{r}\frac{d^2v}{dr^2} - \cancel{2\frac{v}{r^3}} + \cancel{\frac{2}{r^2}\frac{dv}{dr}} = k^2\frac{v}{r}.$$

Multiply both sides by r.

$$\frac{d^2v}{dr^2} = k^2 v$$

The general solution can be written in terms of exponential functions.

$$v(r) = Ae^{kr} + Be^{-kr}$$

Therefore,

$$u(r) = \frac{Ae^{kr} + Be^{-kr}}{r}.$$