Exercise 5

Solve $u_{xx} + u_{yy} = 1$ in $r < a$ with $u(x, y)$ vanishing on $r = a$.

Solution

The PDE we have to solve is known as the Poisson equation.

$$\nabla^2 u = 1$$

Since the domain we want to solve it in is a circle ($r < a$), we opt to write the Laplacian operator in polar coordinates.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 1$$

Because $u = 0$ on the boundary and not some function of θ, we assume that the solution is radially symmetric, that is, it only depends on r, $u = u(r)$. Consequently, the PDE simplifies to an ODE that can be solved relatively easily.

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} = 1$$

Notice that this is a first-order ODE for du/dr. Multiply both sides by the integrating factor

$$I = \exp\left(\int^r \frac{1}{s}\, ds\right) = \exp(\ln r) = r$$

to get

$$r\frac{d^2 u}{dr^2} + \frac{du}{dr} = r.$$

The left side can be written as $d/dr(I du/dr)$ as a result of the product rule.

$$\frac{d}{dr}\left(r\frac{du}{dr}\right) = r$$

Integrate both sides with respect to r.

$$r\frac{du}{dr} = \frac{r^2}{2} + C_1$$

Divide both sides by r.

$$\frac{du}{dr} = \frac{r}{2} + \frac{C_1}{r}$$

Integrate both sides with respect to r once more.

$$u(r) = \frac{r^2}{4} + C_1 \ln r + C_2$$

Apply the boundary conditions here to determine the constants, C_1 and C_2, namely $u(a) = 0$ and $u(0) = \text{finite}$. In order to satisfy the second one, we require $C_1 = 0$.

$$u(a) = \frac{a^2}{4} + C_2 = 0 \quad \rightarrow \quad C_2 = -\frac{a^2}{4}$$

Therefore,

$$u(r) = \frac{r^2}{4} - \frac{a^2}{4} \quad \text{or} \quad u(x, y) = \frac{x^2 + y^2 - a^2}{4}.$$

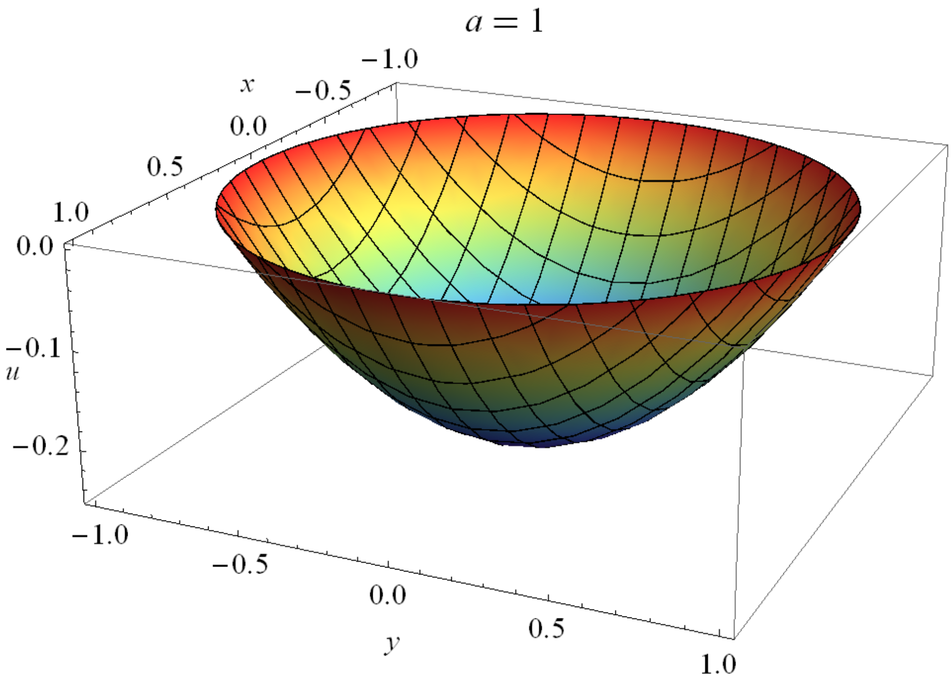

Figure 1: This is a plot of the two-dimensional solution surface $u(x, y)$ in three-dimensional xyu-space for $a = 1$.