Exercise 6

Solve $u_{xx} + u_{yy} = 1$ in the annulus $a < r < b$ with $u(x, y)$ vanishing on both parts of the boundary $r = a$ and $r = b$.

Solution

The PDE we have to solve is known as the Poisson equation.

$$\nabla^2 u = 1$$

Since the domain we want to solve it in is an annulus ($a < r < b$), we opt to write the Laplacian operator in polar coordinates.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 1$$

Because $u = 0$ on the boundaries and not some functions of θ, we assume that the solution is radially symmetric, that is, it only depends on r, $u = u(r)$. Consequently, the PDE simplifies to an ODE that can be solved relatively easily.

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} = 1$$

Notice that this is a first-order ODE for du/dr. Multiply both sides by the integrating factor

$$I = \exp\left(\int^r \frac{1}{s}\,ds\right) = \exp(\ln r) = r$$

to get

$$r\frac{d^2 u}{dr^2} + \frac{du}{dr} = r.$$

The left side can be written as $d/dr(I du/dr)$ as a result of the product rule.

$$\frac{d}{dr}\left(r\frac{du}{dr}\right) = r$$

Integrate both sides with respect to r.

$$r\frac{du}{dr} = \frac{r^2}{2} + C_1$$

Divide both sides by r.

$$\frac{du}{dr} = \frac{r}{2} + \frac{C_1}{r}$$

Integrate both sides with respect to r once more.

$$u(r) = \frac{r^2}{4} + C_1 \ln r + C_2$$

Apply the boundary conditions here to determine the constants, C_1 and C_2.

$$u(a) = \frac{a^2}{4} + C_1 \ln a + C_2 = 0$$
$$u(b) = \frac{b^2}{4} + C_1 \ln b + C_2 = 0$$

This is a system of two equations for two unknowns. Solving it gives

$$C_1 = \frac{a^2 - b^2}{4\ln\frac{b}{a}} \quad \text{and} \quad C_2 = \frac{b^2\ln a - a^2\ln b}{4\ln\frac{b}{a}}.$$

So then

$$\begin{aligned} u(r) &= \frac{r^2}{4} + \frac{a^2 - b^2}{4\ln\frac{b}{a}}\ln r + \frac{b^2\ln a - a^2\ln b}{4\ln\frac{b}{a}} \\ &= \frac{r^2}{4} + \frac{a^2\ln r - b^2\ln r + b^2\ln a - a^2\ln b}{4\ln\frac{b}{a}} \\ &= \frac{r^2}{4} + \frac{-a^2(\ln b - \ln r) - b^2(\ln r - \ln a)}{4\ln\frac{b}{a}} \end{aligned}$$

Therefore,

$$u(r) = \frac{1}{4}\left(r^2 - \frac{a^2\ln\frac{b}{r} + b^2\ln\frac{r}{a}}{\ln\frac{b}{a}}\right)$$

or

$$u(x,y) = \frac{1}{4}\left(x^2 + y^2 - \frac{a^2\ln\frac{b}{\sqrt{x^2+y^2}} + b^2\ln\frac{\sqrt{x^2+y^2}}{a}}{\ln\frac{b}{a}}\right).$$

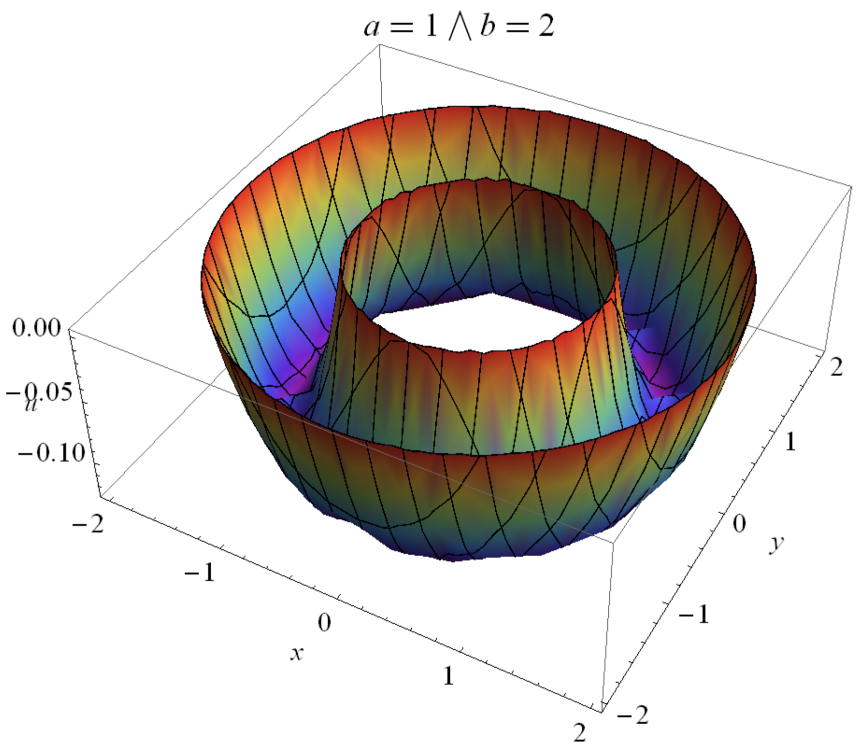

Figure 1: This is a plot of the two-dimensional solution surface $u(x,y)$ in three-dimensional xyu-space for $a = 1$ and $b = 2$.