Exercise 7

Solve $u_{xx} + u_{yy} + u_{zz} = 1$ in the spherical shell $a < r < b$ with $u(x, y, z)$ vanishing on both the inner and outer boundaries.

Solution

The PDE we have to solve is known as the Poisson equation.

$$\nabla^2 u = 1$$

Since the region we're solving it in is a spherical shell, we will expand the Laplacian operator in spherical coordinates (θ here represents the angle from the polar axis).

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2 u}{\partial \theta^2} + (\cot\theta)\frac{\partial u}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\right] = 1$$

We assume from the boundary conditions that the solution is spherically symmetric, that is, it only depends on r, $u = u(r)$. Consequently, the PDE simplifies to an ODE that can be solved relatively easily.

$$\frac{d^2 u}{dr^2} + \frac{2}{r}\frac{du}{dr} = 1$$

Notice that this is a first-order ODE for du/dr. Multiply both sides by the integrating factor

$$I = \exp\left(\int^r \frac{2}{s}\,ds\right) = \exp(2\ln r) = \exp(\ln r^2) = r^2$$

to get

$$r^2\frac{d^2 u}{dr^2} + 2r\frac{du}{dr} = r^2.$$

The left side can be written as $d/dr(I du/dr)$ as a result of the product rule.

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = r^2$$

Integrate both sides with respect to r.

$$r^2\frac{du}{dr} = \frac{r^3}{3} + C_1$$

Divide both sides by r^2.

$$\frac{du}{dr} = \frac{r}{3} + \frac{C_1}{r^2}$$

Integrate both sides with respect to r once more.

$$u(r) = \frac{r^2}{6} - \frac{C_1}{r} + C_2$$

Apply the boundary conditions here to determine the constants, C_1 and C_2.

$$u(a) = \frac{a^2}{6} - \frac{C_1}{a} + C_2 = 0$$

$$u(b) = \frac{b^2}{6} - \frac{C_1}{b} + C_2 = 0$$

This is a system of two equations for two unknowns. Solving it gives

$$C_1 = -\frac{1}{6}ab(a+b) \quad \text{and} \quad C_2 = -\frac{1}{6}(a^2+ab+b^2).$$

So then

$$\begin{aligned} u(r) &= \frac{r^2}{6} + \frac{1}{6r}ab(a+b) - \frac{1}{6}(a^2+ab+b^2) \\ &= \frac{r^3 + ab(a+b) - r(a^2+ab+b^2)}{6r} \\ &= \frac{r^3 + a^2b + ab^2 - a^2r - abr - b^2r}{6r}. \end{aligned}$$

Therefore,

$$u(r) = -\frac{(r-a)(b-r)(r+a+b)}{6r}$$

or

$$u(x,y,z) = -\frac{(\sqrt{x^2+y^2+z^2}-a)(b-\sqrt{x^2+y^2+z^2})(\sqrt{x^2+y^2+z^2}+a+b)}{6\sqrt{x^2+y^2+z^2}}.$$

Note that any of the previous forms for u are acceptable answers. The advantage of the last two is that it's easy to see the boundary conditions are satisfied.