Exercise 9

A spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution. Its inner boundary is held at 100°C. Its outer boundary satisfies $\partial u/\partial r = -\gamma < 0$, where γ is a constant.

(a) Find the temperature. (*Hint:* The temperature depends only on the radius.)

(b) What are the hottest and coldest temperatures?

(c) Can you choose γ so that the temperature on its outer boundary is 20°C?

Solution

Part (a)

The governing equation for the steady-state temperature in an object with no heat source is the Laplace equation.

$$\nabla^2 u = 0$$

Since the object is a spherical shell, we choose to expand the Laplacian operator in spherical coordinates.

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2 u}{\partial \theta^2} + (\cot\theta)\frac{\partial u}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\right] = 0$$

The temperature is prescribed at the inner radius, so the first boundary condition is $u = 100$ at $r = 1$. The heat flux is prescribed at the outer radius, so the second boundary condition is $\partial u/\partial r = -\gamma$ at $r = 2$. Since neither boundary condition depends on θ, we assume that the solution is spherically symmetric, that is, u only depends on r, $u = u(r)$. Consequently, the PDE simplifies to an ODE that can be solved relatively easily.

$$\frac{d^2 u}{dr^2} + \frac{2}{r}\frac{du}{dr} = 0$$

Notice that this is a first-order ODE for du/dr. Multiply both sides by the integrating factor

$$I = \exp\left(\int^r \frac{2}{s}\,ds\right) = \exp(2\ln r) = \exp(\ln r^2) = r^2$$

to get

$$r^2\frac{d^2 u}{dr^2} + 2r\frac{du}{dr} = 0.$$

The left side can be written as $d/dr(I du/dr)$ as a result of the product rule.

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 0$$

Integrate both sides with respect to r.

$$r^2\frac{du}{dr} = C_1$$

Divide both sides by r^2.

$$\frac{du}{dr} = \frac{C_1}{r^2}$$

Apply the boundary condition at the outer radius now to determine C_1.

$$\frac{du}{dr}(2) = \frac{C_1}{4} = -\gamma \quad \rightarrow \quad C_1 = -4\gamma$$

The formula for du/dr becomes

$$\frac{du}{dr} = \frac{-4\gamma}{r^2}.$$

Integrate both sides with respect to r once more.

$$u(r) = \frac{4\gamma}{r} + C_2$$

Apply the boundary condition at the inner radius now to determine C_2.

$$u(1) = 4\gamma + C_2 = 100 \quad \rightarrow \quad C_2 = 100 - 4\gamma$$

So then

$$u(r) = \frac{4\gamma}{r} + 100 - 4\gamma.$$

Therefore, the steady-state temperature distribution is

$$u(r) = 100 - 4\gamma\left(1 - \frac{1}{r}\right), \quad 1 < r < 2.$$

Part (b)

Since du/dr is never zero, the maximum and minimum temperatures occur at the ends of the domain, $1 < r < 2$, by the extreme value theorem.

$$\begin{aligned} &\text{Hottest Temperature:} \quad u(1) = 100 \\ &\text{Coldest Temperature:} \quad u(2) = 100 - 2\gamma \end{aligned}$$

This is consistent with the maximum principle for the Laplace equation, which states that the maximum and minimum values of u must occur on the boundary.

Part (c)

Yes, because the temperature is in terms of γ.

$$u(2) = 100 - 2\gamma = 20 \quad \rightarrow \quad \gamma = 40$$