Exercise 5

Solve Example 1 in the case $b = 1$, $g(x) = h(x) = k(x) = 0$ but $j(x)$ an arbitrary function.

[TYPO: k and j are functions of y, not x (see Figure 1 from page 162).]

Solution

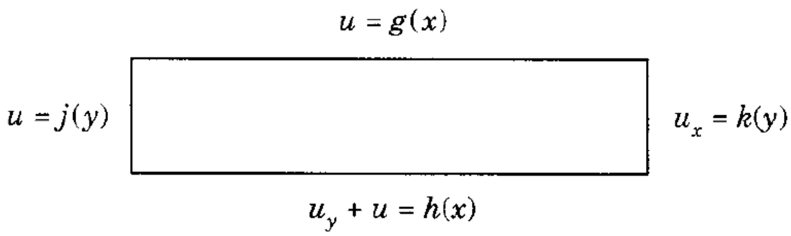


Figure 1

In Example 1 the Laplace equation is solved over the boundary shown in Figure 1 with $j(y) = h(x) = k(y) = 0$. We have to solve the following boundary value problem in this exercise.

$$\nabla^2 u = 0, \quad 0 < x < a, \; 0 < y < 1$$
$$u_y(x,0) + u(x,0) = 0, \quad u(0,y) = j(y)$$
$$u(x,1) = 0, \quad u_x(a,y) = 0$$

As all but one of the boundary conditions are homogeneous, the method of separation of variables can be applied to solve the PDE. Assume a product solution of the form $u = X(x)Y(y)$ and plug it into the PDE

$$u_{xx} + u_{yy} = 0 \quad \rightarrow \quad X''Y + XY'' = 0$$

and the homogeneous boundary conditions.

$$\begin{aligned} u_y(x,0) + u(x,0) = 0 \quad &\rightarrow \quad X(x)Y'(0) + X(x)Y(0) = 0 \quad \rightarrow \quad Y'(0) + Y(0) = 0 \\ u(x,1) = 0 \quad &\rightarrow \quad X(x)Y(1) = 0 \quad \rightarrow \quad Y(1) = 0 \\ u_x(a,y) = 0 \quad &\rightarrow \quad X'(a)Y(y) = 0 \quad \rightarrow \quad X'(a) = 0 \end{aligned}$$

Now separate variables in the PDE: bring all functions of x to the left side and all functions of y to the right side. Note that the final answer will be the same regardless of which side the minus sign is on.

$$X''Y + XY'' = 0 \quad \rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y}$$

The only way that a function of x can be equal to a function of y is if both are equal to a constant λ.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

Values of λ for which the boundary conditions are satisfied are called the eigenvalues, and the nontrivial functions associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

If λ is positive, then the ODE for Y becomes

$$-\frac{Y''}{Y} = \mu^2.$$

Multiply both sides by $-Y$.

$$Y'' = -\mu^2 Y$$

The general solution can be written in terms of sine and cosine.

$$Y(y) = C_1 \cos \mu y + C_2 \sin \mu y$$

Take a derivative of it with respect to y.

$$Y'(y) = \mu(-C_1 \sin \mu y + C_2 \cos \mu y)$$

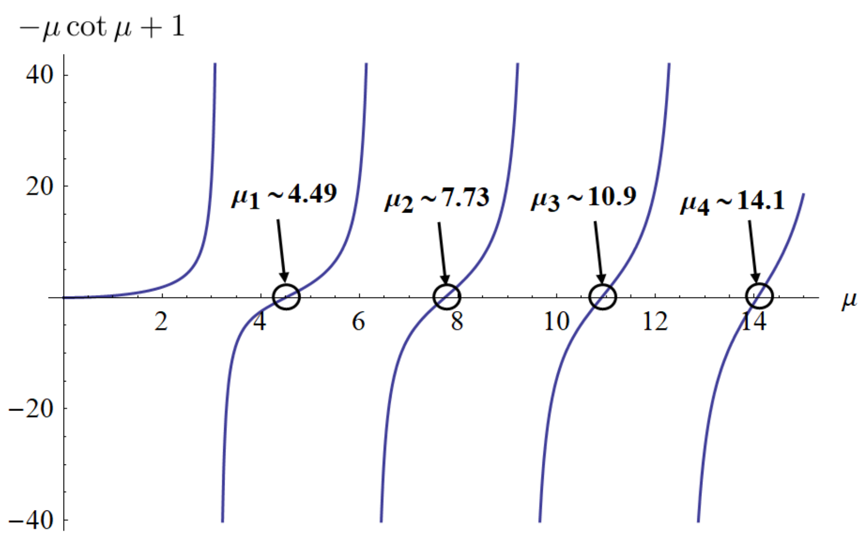
Apply the two boundary conditions for Y to determine C_1 and C_2.

$$\begin{aligned} Y'(0) + Y(0) = \mu(C_2) + C_1 = 0 \\ Y(1) = C_1 \cos \mu + C_2 \sin \mu = 0 \end{aligned}$$

From the first equation we have $C_1 = -\mu C_2$, so the second equation becomes

$$-\mu C_2 \cos \mu + C_2 \sin \mu = 0 \quad \rightarrow \quad -\mu \cot \mu + 1 = 0.$$

Hence, the positive eigenvalues are $\lambda_n = \mu_n^2$, $n = 1, 2, \ldots$, where μ_n is the nth zero of $-\mu \cot \mu + 1$.



The eigenfunctions associated with these eigenvalues are

$$\begin{aligned} Y(y) &= C_1 \cos \mu y + C_2 \sin \mu y \\ &= -\mu C_2 \cos \mu y + C_2 \sin \mu y \\ &= -C_2(\mu \cos \mu y - \sin \mu y) \quad \rightarrow \quad Y_n(y) = \mu_n \cos \mu_n y - \sin \mu_n y, \quad n = 1, 2, \ldots. \end{aligned}$$

Since $Y(y)$ is relevant, the related ODE for $X(x)$ will now be solved.

$$\frac{X''}{X} = \mu^2$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_3 \cosh \mu x + C_4 \sinh \mu x$$

Take a derivative of this solution with respect to x.

$$X'(x) = \mu(C_3 \sinh \mu x + C_4 \cosh \mu x)$$

Apply the boundary condition at $x = a$ to determine one of the constants.

$$X'(a) = \mu(C_3 \sinh \mu a + C_4 \cosh \mu a) = 0 \quad \rightarrow \quad C_4 = -C_3 \frac{\sinh \mu a}{\cosh \mu a}$$

So then

$$\begin{aligned}
X(x) &= C_3 \cosh \mu x - C_3 \frac{\sinh \mu a}{\cosh \mu a} \sinh \mu x \\
&= C_3 \left(\cosh \mu x - \frac{\sinh \mu a}{\cosh \mu a} \sinh \mu x \right) \\
&= C_3 \frac{\cosh \mu a \cosh \mu x - \sinh \mu a \sinh \mu x}{\cosh \mu a} \\
&= C_3 \frac{\cosh(\mu a - \mu x)}{\cosh \mu a} \\
&= C_5 \cosh[\mu(a - x)] \quad \rightarrow \quad X_n(x) = \cosh[\mu_n(a - x)], \quad n = 1, 2, \ldots.
\end{aligned}$$

Determination of the Zero Eigenvalue: $\lambda = 0$

If λ is zero, then the ODE for Y becomes

$$-\frac{Y''}{Y} = 0.$$

Multiply both sides by $-Y$.

$$Y''(y) = 0$$

Integrate both sides with respect to y.

$$Y'(y) = C_6$$

Integrate both sides with respect to y once more.

$$Y(y) = C_6 y + C_7$$

Apply the boundary conditions here to determine C_6 and C_7.

$$\begin{aligned}
Y'(0) + Y(0) &= C_6 + C_7 = 0 \\
Y(1) &= C_6 + C_7 = 0
\end{aligned}$$

Both conditions lead to $C_6 = -C_7$.

$$\begin{aligned} Y(y) &= -C_7 y + C_7 \\ &= C_7(1 - y) \end{aligned}$$

We find that zero is an eigenvalue. The related ODE for $X(x)$ will now be solved.

$$\frac{X''}{X} = 0$$

Multiply both sides by X.

$$X''(x) = 0$$

Integrate both sides with respect to x.

$$X'(x) = C_8$$

Apply the boundary condition at $x = a$ to find C_8.

$$X'(a) = C_8 = 0$$

So then

$$X'(x) = 0.$$

Integrate both sides with respect to x once more.

$$X(x) = C_9$$

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

If λ is negative, then the ODE for Y becomes

$$-\frac{Y''}{Y} = -\gamma^2.$$

Multiply both sides by $-Y$.

$$Y'' = \gamma^2 Y$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_{10} \cosh \gamma y + C_{11} \sinh \gamma y$$

Take a derivative of it with respect to y.

$$Y'(y) = \gamma(C_{10} \sinh \gamma y + C_{11} \cosh \gamma y)$$

Apply the two boundary conditions for Y to determine C_{10} and C_{11}.

$$\begin{aligned} Y'(0) + Y(0) = \gamma(C_{11}) + C_{10} = 0 \\ Y(1) = C_{10} \cosh \gamma + C_{11} \sinh \gamma = 0 \end{aligned}$$

From the first equation we have $C_{10} = -\gamma C_{11}$, so the second equation becomes

$$-\gamma C_{11} \cosh \gamma + C_{11} \sinh \gamma = 0 \quad \to \quad -\gamma \coth \gamma + 1 = 0.$$

There are no negative eigenvalues because there are no values of γ that satisfy $-\gamma \coth \gamma + 1 = 0$ (besides zero).

According to the principle of superposition, the general solution for u is a linear combination of the eigenfunctions $X(x)Y(y)$ over all the eigenvalues.

$$\boxed{u(x,y) = A_0(1-y) + \sum_{n=1}^{\infty} A_n \cosh[\mu_n(a-x)](\mu_n \cos \mu_n y - \sin \mu_n y),}$$

where μ_n is the nth zero of $-\mu \cot \mu + 1$. In order to find A_0 and A_n, we have to use the inhomogeneous boundary condition at $x = 0$.

$$u(0,y) = A_0(1-y) + \sum_{n=1}^{\infty} A_n \cosh \mu_n a(\mu_n \cos \mu_n y - \sin \mu_n y) = j(y) \tag{1}$$

To determine A_0, multiply both sides of equation (1) by $1 - y$

$$A_0(1-y)^2 + \sum_{n=1}^{\infty} A_n \cosh \mu_n a(\mu_n \cos \mu_n y - \sin \mu_n y)(1-y) = j(y)(1-y)$$

and integrate both sides with respect to y from 0 to 1.

$$\int_0^1 \left\{ A_0(1-y)^2 + \sum_{n=1}^{\infty} A_n \cosh \mu_n a(\mu_n \cos \mu_n y - \sin \mu_n y)(1-y) \right\} dy = \int_0^1 j(y)(1-y)\, dy$$

Split up the integral into two and bring the constants in front.

$$A_0 \int_0^1 (1-y)^2\, dy + \sum_{n=1}^{\infty} A_n \cosh \mu_n a \underbrace{\int_0^1 (\mu_n \cos \mu_n y - \sin \mu_n y)(1-y)\, dy}_{=0} = \int_0^1 j(y)(1-y)\, dy$$

Since the boundary conditions are symmetric (Dirichlet, Neumann, or Robin), the eigenfunctions corresponding to different eigenvalues are guaranteed to be orthogonal. Consequently, the second integral on the left is zero. Evaluate the first one

$$A_0 \cdot \frac{1}{3} = \int_0^1 j(y)(1-y)\, dy$$

and solve for A_0.

$$\boxed{A_0 = 3 \int_0^1 j(y)(1-y)\, dy}$$

To determine A_n, multiply both sides of equation (1) by $\mu_m \cos \mu_m y - \sin \mu_m y$, where m is an integer.

$$A_0(1-y)(\mu_m \cos \mu_m y - \sin \mu_m y) + \sum_{n=1}^{\infty} A_n \cosh \mu_n a(\mu_n \cos \mu_n y - \sin \mu_n y)(\mu_m \cos \mu_m y - \sin \mu_m y)$$
$$= j(y)(\mu_m \cos \mu_m y - \sin \mu_m y)$$

Integrate both sides with respect to y from 0 to 1.

$$\int_0^1 \left\{ A_0(1-y)(\mu_m \cos\mu_m y - \sin\mu_m y) + \sum_{n=1}^{\infty} A_n \cosh\mu_n a(\mu_n \cos\mu_n y - \sin\mu_n y)(\mu_m \cos\mu_m y - \sin\mu_m y) \right\} dy$$
$$= \int_0^1 j(y)(\mu_m \cos\mu_m y - \sin\mu_m y)\, dy$$

Split up the integral on the left into two and bring the constants in front.

$$A_0 \underbrace{\int_0^1 (1-y)(\mu_m \cos\mu_m y - \sin\mu_m y)\, dy}_{=0} + \sum_{n=1}^{\infty} A_n \cosh\mu_n a \int_0^1 (\mu_n \cos\mu_n y - \sin\mu_n y)(\mu_m \cos\mu_m y - \sin\mu_m y)\, dy$$
$$= \int_0^1 j(y)(\mu_m \cos\mu_m y - \sin\mu_m y)\, dy$$

Since the boundary conditions are symmetric (Dirichlet, Neumann, or Robin), the eigenfunctions corresponding to different eigenvalues are guaranteed to be orthogonal. Consequently, the first integral on the left is zero and the second integral on the left is zero for $n \neq m$, which means only one term remains in the infinite series: $n = m$.

$$A_n \cosh\mu_n a \int_0^1 (\mu_n \cos\mu_n y - \sin\mu_n y)^2\, dy = \int_0^1 j(y)(\mu_n \cos\mu_n y - \sin\mu_n y)\, dy$$

Evaluate the integral on the left side.

$$A_n \cosh\mu_n a \cdot \frac{2\mu_n(\mu_n^2 + \cos 2\mu_n) + (\mu_n^2 - 1)\sin 2\mu_n}{4\mu_n} = \int_0^1 j(y)(\mu_n \cos\mu_n y - \sin\mu_n y)\, dy$$

Now we can solve for A_n.

$$\boxed{A_n = \frac{4\mu_n \operatorname{sech}\mu_n a}{2\mu_n(\mu_n^2 + \cos 2\mu_n) + (\mu_n^2 - 1)\sin 2\mu_n} \int_0^1 j(y)(\mu_n \cos\mu_n y - \sin\mu_n y)\, dy, \quad n = 1, 2, \ldots}$$