Exercise 6

Solve the following Neumann problem in the cube $\{0 < x < 1,\ 0 < y < 1,\ 0 < z < 1\}$: $\Delta u = 0$ with $u_z(x, y, 1) = g(x, y)$ and homogeneous Neumann conditions on the other five faces, where $g(x, y)$ is an arbitrary function with zero average.

Solution[1]

The boundary value problem to solve is the following. ($\Delta = \nabla^2$)

$$\begin{gathered}\nabla^2 u = 0, \quad 0 < x < 1,\ 0 < y < 1,\ 0 < z < 1 \\ u_x(0, y, z) = 0, \quad u_y(x, 0, z) = 0, \quad u_z(x, y, 0) = 0 \\ u_x(1, y, z) = 0, \quad u_y(x, 1, z) = 0, \quad u_z(x, y, 1) = g(x, y)\end{gathered}$$

As all but one of the boundary conditions are homogeneous, the method of separation of variables can be applied to solve the PDE. Assume a product solution of the form $u = X(x)Y(y)Z(z)$ and plug it into the PDE

$$u_{xx} + u_{yy} + u_{zz} = 0 \quad \rightarrow \quad X''YZ + XY''Z + XYZ'' = 0$$

and the homogeneous boundary conditions.

$$\begin{aligned}
u_x(0, y, z) = 0 \quad &\rightarrow \quad X'(0)Y(y)Z(z) = 0 \quad &\rightarrow \quad X'(0) = 0 \\
u_x(1, y, z) = 0 \quad &\rightarrow \quad X'(1)Y(y)Z(z) = 0 \quad &\rightarrow \quad X'(1) = 0 \\
u_y(x, 0, z) = 0 \quad &\rightarrow \quad X(x)Y'(0)Z(z) = 0 \quad &\rightarrow \quad Y'(0) = 0 \\
u_y(x, 1, z) = 0 \quad &\rightarrow \quad X(x)Y'(1)Z(z) = 0 \quad &\rightarrow \quad Y'(1) = 0 \\
u_z(x, y, 0) = 0 \quad &\rightarrow \quad X(x)Y(y)Z'(0) = 0 \quad &\rightarrow \quad Z'(0) = 0
\end{aligned}$$

Now separate variables in the PDE: bring all functions of x to the left side and all functions of y and z to the right side. The final answer will be the same regardless of which side the minus sign is on.

$$X''YZ + XY''Z + XYZ'' = 0 \quad \rightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0 \quad \rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z}$$

The only way that a function of x can be equal to a function of y and z is if both are equal to a constant λ.

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = \lambda$$

Bring Y''/Y to the right side and λ to the left.

$$-\lambda - \frac{Z''}{Z} = \frac{Y''}{Y}$$

The only way a function of z can be equal to a function of y is if both sides are equal to another constant η.

$$-\lambda - \frac{Z''}{Z} = \frac{Y''}{Y} = \eta$$

[1] Thank you, Gary, for letting me know I messed up here. The solution's been rewritten; I'm sorry it took so long.

To summarize, using the method of separation of variables reduces the Laplace equation to three ODEs, one in each spatial variable.

$$\left.\begin{aligned} \frac{X''}{X} &= \lambda \\ \frac{Y''}{Y} &= \eta \\ -\lambda - \frac{Z''}{Z} &= \eta \end{aligned}\right\}$$

Values of λ and η for which the boundary conditions are satisfied are known as the eigenvalues, and the functions associated with them are known as the eigenfunctions. Start by solving the ODE for X.

$$X'' = \lambda X$$

Check to see whether there are any positive eigenvalues: $\lambda = \mu^2$.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Differentiate it with respect to x.

$$X'(x) = \mu(C_1 \sinh \mu x + C_2 \cosh \mu x)$$

Apply the boundary conditions, $X'(0) = 0$ and $X'(1) = 0$, to determine C_1 and C_2.

$$\begin{aligned} X'(0) &= \mu(C_2) = 0 \\ X'(1) &= \mu(C_1 \sinh \mu + C_2 \cosh \mu) = 0 \end{aligned}$$

The first equation gives $C_2 = 0$, which makes the second equation $C_1 \mu \sinh \mu = 0$. No positive value of μ can satisfy it, which means there are no positive eigenvalues. Now check to see if zero is an eigenvalue.

$$X'' = 0$$

The general solution is a straight line.

$$X(x) = C_3 x + C_4$$

Differentiate it with respect to x.

$$X'(x) = C_3$$

Set $C_3 = 0$ to satisfy the boundary conditions.

$$X(x) = C_4$$

This is not the trivial solution, so zero is an eigenvalue. The eigenfunction associated with it is a constant. With $\lambda = 0$, solve the remaining eigenvalue problems for Y and Z now.

$$Y'' = \nu Y$$

Check to see if there are any positive eigenvalues by setting $\nu = \alpha^2$.

$$Y'' = \alpha^2 Y$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_5 \cosh \alpha y + C_6 \sinh \alpha y$$

Differentiate it with respect to y.

$$Y'(y) = \alpha(C_5 \sinh \alpha y + C_6 \cosh \alpha y)$$

Apply the boundary conditions, $Y(0) = 0$ and $Y(1) = 0$, to determine C_5 and C_6.

$$Y'(0) = \alpha(C_6) = 0$$
$$Y'(1) = \alpha(C_5 \sinh \alpha + C_6 \cosh \alpha) = 0$$

The first equation gives $C_6 = 0$, which makes the second equation $C_5\alpha \sinh \alpha = 0$. No positive value of α can satisfy it, which means there are no positive eigenvalues. Now check to see if zero is an eigenvalue.

$$Y'' = 0$$

The general solution is a line.

$$Y(y) = C_7 y + C_8$$

Differentiate it with respect to y.

$$Y'(y) = C_7$$

Set $C_7 = 0$ to satisfy the boundary conditions.

$$Y(y) = C_8$$

This is not the trivial solution, so zero is an eigenvalue. The eigenfunction associated with it is a constant. With $\lambda = 0$ and $\eta = 0$, solve the remaining eigenvalue problem for Z now.

$$-\frac{Z''}{Z} = 0$$

$$Z'' = 0$$

The general solution is a straight line.

$$Z(z) = C_9 z + C_{10}$$

Differentiate it with respect to z.

$$Z'(z) = C_9$$

Set $C_9 = 0$ to satisfy the $Z'(0) = 0$ boundary condition. As a result, $\lambda = 0$ and $\eta = 0$ lead to a constant eigenfunction for $Z(z)$. There will be a constant term in the solution for u. With $\lambda = 0$, go back and check the eigenvalue problem of Y for negative eigenvalues: $\nu = -\beta^2$.

$$Y'' = -\beta^2 Y$$

The general solution can be written in terms of sine and cosine.

$$Y(y) = C_{11} \cos \beta y + C_{12} \sin \beta y$$

Differentiate it with respect to y.

$$Y'(y) = \beta(-C_{11} \sin \beta y + C_{12} \cos \beta y)$$

Apply the boundary conditions, $Y(0) = 0$ and $Y'(1) = 0$, to determine C_{11} and C_{12}.

$$\begin{aligned} Y'(0) &= \beta(C_{12}) = 0 \\ Y'(1) &= \beta(-C_{11}\sin\beta + C_{12}\cos\beta) = 0 \end{aligned}$$

This first equation gives $C_{12} = 0$, which makes the second equation $-C_{11}\beta\sin\beta = 0$.

$$\sin\beta = 0$$

$$\beta = n\pi, \quad n = 1, 2, \ldots$$

There are negative eigenvalues $\eta = -(n\pi)^2$, and the eigenfunctions associated with them are

$$Y(y) = C_{11}\cos\beta y \quad \rightarrow \quad Y_n(y) = \cos n\pi y.$$

With $\lambda = 0$ and $\eta = -n^2\pi^2$, solve the remaining eigenvalue problem for Z.

$$-\frac{Z''}{Z} = -n^2\pi^2$$

$$Z'' = n^2\pi^2 Z$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$Z(z) = C_{13}\cosh n\pi z + C_{14}\sinh n\pi z$$

Differentiate it with respect to z.

$$Z'(z) = C_{13}n\pi\sinh n\pi z + C_{14}n\pi\cosh n\pi z$$

Apply the boundary condition $Z'(0) = 0$ to determine one of the constants.

$$Z'(0) = C_{14}n\pi = 0 \quad \rightarrow \quad C_{14} = 0$$

As a result, $\lambda = 0$ and $\eta = -n^2\pi^2$ lead to an eigenfunction $\cosh n\pi z$ for Z. There will be a term containing $\cos n\pi y\cosh n\pi z$ in the solution for u. Now go back to the eigenvalue problem for X and check to see if it has negative eigenvalues: $\lambda = -\gamma^2$.

$$X'' = -\gamma^2 X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_{15}\cos\gamma x + C_{16}\sin\gamma x$$

Differentiate it with respect to x.

$$X'(x) = \gamma(-C_{15}\sin\gamma x + C_{16}\cos\gamma x)$$

Apply the boundary conditions, $X(0) = 0$ and $X(1) = 0$, to determine C_{15} and C_{16}.

$$\begin{aligned} X'(0) &= \gamma(C_{16}) = 0 \\ X'(1) &= \gamma(-C_{15}\sin\gamma + C_{16}\cos\gamma) = 0 \end{aligned}$$

This first equation gives $C_{16} = 0$, which makes the second equation become $-C_{15}\gamma \sin\gamma = 0$.

$$\sin\gamma = 0$$

$$\gamma = n\pi, \quad n = 1, 2, \ldots$$

There are negative eigenvalues $\lambda = -(n\pi)^2$, and the eigenfunctions associated with them are

$$X(x) = C_{15}\cos\gamma x \quad \to \quad X_n(x) = \cos n\pi x.$$

With $\lambda = -(n\pi)^2$, solve the remaining eigenvalue problems for Y and Z now.

$$Y'' = \nu Y$$

No positive eigenvalues were found, and zero was found to be an eigenvalue with the associated eigenfunction being a constant. With $\lambda = -n^2\pi^2$ and $\eta = 0$, solve the eigenvalue problem for Z.

$$n^2\pi^2 - \frac{Z''}{Z} = 0$$

$$Z'' = n^2\pi^2 Z$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$Z(z) = C_{17}\cosh n\pi z + C_{18}\sinh n\pi z$$

Take a derivative with respect to z.

$$Z'(z) = C_{17}n\pi\sinh n\pi z + C_{18}n\pi\cosh n\pi z$$

Apply the boundary condition $Z'(0) = 0$ to determine one of the constants.

$$Z'(0) = C_{18}n\pi = 0 \quad \to \quad C_{18} = 0$$

As a result, $\lambda = -n^2\pi^2$ and $\eta = 0$ lead to an eigenfunction $\cosh n\pi z$ for Z. There will be a term containing $\cos n\pi x \cosh n\pi z$ in the solution for u. Finally, with $\lambda = -n^2\pi^2$, check the eigenvalue problem of Y for negative eigenvalues ($\eta = -\beta^2$) and then solve the eigenvalue problem for Z with them.

$$Y'' = -\beta^2 Y$$

The general solution can be written in terms of sine and cosine.

$$Y(y) = C_{19}\cos\beta y + C_{20}\sin\beta y$$

Differentiate it with respect to y.

$$Y'(y) = \beta(-C_{19}\sin\beta y + C_{20}\cos\beta y)$$

Apply the boundary conditions, $Y(0) = 0$ and $Y(1) = 0$, to determine C_{19} and C_{20}.

$$\begin{aligned} Y'(0) &= \beta(C_{20}) = 0 \\ Y'(1) &= \beta(-C_{19}\sin\beta + C_{20}\cos\beta) = 0 \end{aligned}$$

This first equation gives $C_{20} = 0$, which means the second equation becomes $-C_{19}\beta \sin \beta = 0$.

$$\sin \beta = 0$$

$$\beta = m\pi, \quad m = 1, 2, \ldots$$

There are negative eigenvalues $\eta = -(m\pi)^2$, and the eigenfunctions associated with them are

$$Y(y) = C_{19} \cos \beta y \quad \to \quad Y_m(y) = \cos m\pi y.$$

With $\lambda = -n^2\pi^2$ and $\eta = -m^2\pi^2$, solve the eigenvalue problem for Z.

$$n^2\pi^2 - \frac{Z''}{Z} = -m^2\pi^2$$

$$Z'' = (n^2 + m^2)\pi^2 Z$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$Z(z) = C_{21} \cosh(\sqrt{n^2 + m^2}\pi z) + C_{22} \sinh(\sqrt{n^2 + m^2}\pi z)$$

Differentiate it with respect to z.

$$Z'(z) = \sqrt{n^2 + m^2}\pi \{C_{21} \sinh(\sqrt{n^2 + m^2}\pi z) + C_{22} \cosh(\sqrt{n^2 + m^2}\pi z)\}$$

Apply the boundary condition $Z'(0) = 0$ to determine one of the constants.

$$Z'(0) = \sqrt{n^2 + m^2}\pi (C_{22}) = 0 \quad \to \quad C_{22} = 0$$

As a result, $\lambda = -n^2\pi^2$ and $\eta = -m^2\pi^2$ lead to an eigenfunction $\cosh(\sqrt{n^2 + m^2}\pi z)$ for Z. There will be a term containing $\cos n\pi x \cos m\pi y \cosh(\sqrt{n^2 + m^2}\pi z)$ in the solution for u. According to the principle of superposition, the general solution to the PDE is a linear combination of all the eigenfunctions over all the eigenvalues.

$$\boxed{\begin{aligned} u(x, y, z) = A_0 &+ \sum_{n=1}^{\infty} B_n \cos n\pi y \cosh n\pi z + \sum_{n=1}^{\infty} D_n \cos n\pi x \cosh n\pi z \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_{mn} \cos n\pi x \cos m\pi y \cosh(\sqrt{n^2 + m^2}\pi z) \end{aligned}}$$

Differentiate it with respect to z.

$$\begin{aligned} \frac{\partial u}{\partial z} = &\sum_{n=1}^{\infty} n\pi B_n \cos n\pi y \sinh n\pi z + \sum_{n=1}^{\infty} n\pi D_n \cos n\pi x \sinh n\pi z \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sqrt{n^2 + m^2}\pi E_{mn} \cos n\pi x \cos m\pi y \sinh(\sqrt{n^2 + m^2}\pi z) \end{aligned}$$

Evaluate it at $z = 1$ and use the boundary condition $u_z(x, y, 1) = g(x, y)$.

$$\begin{aligned} \left.\frac{\partial u}{\partial z}\right|_{z=1} = &\sum_{n=1}^{\infty} n\pi B_n \cos n\pi y \sinh n\pi + \sum_{n=1}^{\infty} n\pi D_n \cos n\pi x \sinh n\pi \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sqrt{n^2 + m^2}\pi E_{mn} \cos n\pi x \cos m\pi y \sinh(\sqrt{n^2 + m^2}\pi) = g(x, y) \end{aligned} \tag{1}$$

To find B_n, multiply both sides of equation (1) by $\cos p\pi y$, where p is an integer.

$$\sum_{n=1}^{\infty} n\pi B_n \cos n\pi y \cos p\pi y \sinh n\pi + \sum_{n=1}^{\infty} n\pi D_n \cos n\pi x \cos p\pi y \sinh n\pi \\ + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sqrt{n^2+m^2}\pi E_{mn} \cos n\pi x \cos m\pi y \cos p\pi y \sinh(\sqrt{n^2+m^2}\pi) = g(x,y)\cos p\pi y$$

Integrate both sides with respect to x from 0 to 1, and integrate both sides with respect to y from 0 to 1.

$$\int_0^1 \int_0^1 \left[\sum_{n=1}^{\infty} n\pi B_n \cos n\pi y \cos p\pi y \sinh n\pi + \sum_{n=1}^{\infty} n\pi D_n \cos n\pi x \cos p\pi y \sinh n\pi \\ + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sqrt{n^2+m^2}\pi E_{mn} \cos n\pi x \cos m\pi y \cos p\pi y \sinh(\sqrt{n^2+m^2}\pi)\right] dx\, dy \\ = \int_0^1 \int_0^1 g(x,y)\cos p\pi y\, dx\, dy$$

Split up the integrals and bring the constants in front of each.

$$\sum_{n=1}^{\infty} n\pi B_n \sinh n\pi \left(\int_0^1 dx\right) \int_0^1 \cos n\pi y \cos p\pi y\, dy \\ + \sum_{n=1}^{\infty} n\pi D_n \sinh n\pi \left(\int_0^1 \cos n\pi x\, dx\right) \int_0^1 \cos p\pi y\, dy \\ + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sqrt{n^2+m^2}\pi E_{mn} \sinh(\sqrt{n^2+m^2}\pi) \left(\int_0^1 \cos n\pi x\, dx\right) \int_0^1 \cos m\pi y \cos p\pi y\, dy \\ = \int_0^1 \int_0^1 g(x,y)\cos p\pi y\, dx\, dy$$

Evaluate the integrals in dx on the left side.

$$\sum_{n=1}^{\infty} n\pi B_n \sinh n\pi \int_0^1 \cos n\pi y \cos p\pi y\, dy = \int_0^1 \int_0^1 g(x,y)\cos p\pi y\, dx\, dy$$

Because the cosine functions are orthogonal, the integral on the left is zero if $n \neq p$. It's nonzero only if $n = p$.

$$n\pi B_n \sinh n\pi \int_0^1 \cos^2 n\pi y\, dy = \int_0^1 \int_0^1 g(x,y)\cos n\pi y\, dx\, dy$$

Evaluate the integral on the left.

$$n\pi B_n \sinh n\pi \left(\frac{1}{2}\right) = \int_0^1 \int_0^1 g(x,y)\cos n\pi y\, dx\, dy$$

Therefore,

$$\boxed{B_n = \frac{2}{n\pi \sinh n\pi} \int_0^1 \int_0^1 g(x,y)\cos n\pi y\, dx\, dy.}$$

To find D_n, multiply both sides of equation (1) by $\cos p\pi x$, where p is an integer.

$$\sum_{n=1}^{\infty} n\pi B_n \cos p\pi x \cos n\pi y \sinh n\pi + \sum_{n=1}^{\infty} n\pi D_n \cos n\pi x \cos p\pi x \sinh n\pi \\ + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sqrt{n^2+m^2}\pi E_{mn} \cos n\pi x \cos p\pi x \cos m\pi y \sinh(\sqrt{n^2+m^2}\pi) = g(x,y)\cos p\pi x$$

Integrate both sides with respect to x from 0 to 1, and integrate both sides with respect to y from 0 to 1.

$$\int_0^1 \int_0^1 \left[\sum_{n=1}^{\infty} n\pi B_n \cos p\pi x \cos n\pi y \sinh n\pi + \sum_{n=1}^{\infty} n\pi D_n \cos n\pi x \cos p\pi x \sinh n\pi \right. \\ \left. + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sqrt{n^2+m^2}\pi E_{mn} \cos n\pi x \cos p\pi x \cos m\pi y \sinh(\sqrt{n^2+m^2}\pi)\right] dx\, dy \\ = \int_0^1 \int_0^1 g(x,y)\cos p\pi x \, dx\, dy$$

Split up the integrals and bring the constants in front of each.

$$\sum_{n=1}^{\infty} n\pi B_n \sinh n\pi \left(\int_0^1 \cos p\pi x\, dx\right) \int_0^1 \cos n\pi y\, dy \\ + \sum_{n=1}^{\infty} n\pi D_n \sinh n\pi \left(\int_0^1 dy\right) \int_0^1 \cos n\pi x \cos p\pi x\, dx \\ + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sqrt{n^2+m^2}\pi E_{mn} \sinh(\sqrt{n^2+m^2}\pi) \left(\int_0^1 \cos n\pi x \cos p\pi x\, dx\right) \int_0^1 \cos m\pi y\, dy \\ = \int_0^1 \int_0^1 g(x,y)\cos p\pi x\, dx\, dy$$

Evaluate the integrals in dy on the left side.

$$\sum_{n=1}^{\infty} n\pi D_n \sinh n\pi \int_0^1 \cos n\pi x \cos p\pi x\, dx = \int_0^1 \int_0^1 g(x,y)\cos p\pi x\, dx\, dy$$

Because the cosine functions are orthogonal, the integral on the left is zero if $n \neq p$. It's nonzero only if $n = p$.

$$n\pi D_n \sinh n\pi \int_0^1 \cos^2 n\pi x\, dx = \int_0^1 \int_0^1 g(x,y)\cos n\pi x\, dx\, dy$$

Evaluate the integral on the left.

$$n\pi D_n \sinh n\pi \left(\frac{1}{2}\right) = \int_0^1 \int_0^1 g(x,y)\cos n\pi x\, dx\, dy$$

Therefore,

$$\boxed{D_n = \frac{2}{n\pi \sinh n\pi} \int_0^1 \int_0^1 g(x,y)\cos n\pi x\, dx\, dy.}$$

To find E_{mn}, multiply both sides of equation (1) by $\cos p\pi x \cos q\pi y$, where p and q are integers.

$$\sum_{n=1}^{\infty} n\pi B_n \cos p\pi x \cos n\pi y \cos q\pi y \sinh n\pi + \sum_{n=1}^{\infty} n\pi D_n \cos n\pi x \cos p\pi x \cos q\pi y \sinh n\pi$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sqrt{n^2+m^2}\pi E_{mn} \cos n\pi x \cos p\pi x \cos m\pi y \cos q\pi y \sinh(\sqrt{n^2+m^2}\pi)$$
$$= g(x,y)\cos p\pi x \cos q\pi y$$

Integrate both sides with respect to x from 0 to 1, and integrate both sides with respect to y from 0 to 1.

$$\int_0^1 \int_0^1 \left[\sum_{n=1}^{\infty} n\pi B_n \cos p\pi x \cos n\pi y \cos q\pi y \sinh n\pi + \sum_{n=1}^{\infty} n\pi D_n \cos n\pi x \cos p\pi x \cos q\pi y \sinh n\pi \right.$$
$$\left. + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sqrt{n^2+m^2}\pi E_{mn} \cos n\pi x \cos p\pi x \cos m\pi y \cos q\pi y \sinh(\sqrt{n^2+m^2}\pi)\right] dx\, dy$$
$$= \int_0^1 \int_0^1 g(x,y)\cos p\pi x \cos q\pi y \, dx\, dy$$

Split up the integrals and bring the constants in front of each.

$$\sum_{n=1}^{\infty} n\pi B_n \sinh n\pi \left(\int_0^1 \cos p\pi x\, dx\right) \int_0^1 \cos n\pi y \cos q\pi y \, dy$$
$$+ \sum_{n=1}^{\infty} n\pi D_n \sinh n\pi \left(\int_0^1 \cos q\pi y\, dy\right) \int_0^1 \cos n\pi x \cos p\pi x \, dx$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sqrt{n^2+m^2}\pi E_{mn} \sinh(\sqrt{n^2+m^2}\pi) \left(\int_0^1 \cos n\pi x \cos p\pi x\, dx\right) \int_0^1 \cos m\pi y \cos q\pi y\, dy$$
$$= \int_0^1 \int_0^1 g(x,y)\cos p\pi x \cos q\pi y \, dx\, dy$$

Evaluate the integrals in parentheses in the first two terms on the left. They are both zero.

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sqrt{n^2+m^2}\pi E_{mn} \sinh(\sqrt{n^2+m^2}\pi) \left(\int_0^1 \cos n\pi x \cos p\pi x\, dx\right) \int_0^1 \cos m\pi y \cos q\pi y\, dy$$
$$= \int_0^1 \int_0^1 g(x,y)\cos p\pi x \cos q\pi y \, dx\, dy$$

Because the cosine functions are orthogonal, the integrals on the left are zero if $n \neq p$ and $m \neq q$. They're nonzero only if $n = p$ and $m = q$.

$$\sqrt{n^2+m^2}\pi E_{mn} \sinh(\sqrt{n^2+m^2}\pi) \left(\int_0^1 \cos^2 n\pi x\, dx\right) \int_0^1 \cos^2 m\pi y\, dy$$
$$= \int_0^1 \int_0^1 g(x,y)\cos n\pi x \cos m\pi y \, dx\, dy$$

Evaluate the integrals on the left.

$$\sqrt{n^2+m^2}\pi E_{mn}\sinh(\sqrt{n^2+m^2}\pi)\left(\frac{1}{2}\right)\frac{1}{2} = \int_0^1\int_0^1 g(x,y)\cos n\pi x\cos m\pi y\,dx\,dy$$

Therefore,

$$\boxed{E_{mn} = \frac{4}{\sqrt{n^2+m^2}\pi\sinh(\sqrt{n^2+m^2}\pi)}\int_0^1\int_0^1 g(x,y)\cos n\pi x\cos m\pi y\,dx\,dy.}$$

Note that A_0 remains arbitrary and that g must have an average of zero for there to be a solution.

$$\boxed{\int_0^1\int_0^1 g(x,y)\,dx\,dy = 0}$$

This is obtained by integrating both sides of equation (1) with respect to x and y from 0 to 1.