Exercise 7

(a) Find the harmonic function in the semi-infinite strip $\{0 \le x \le \pi,\ 0 \le y < \infty\}$ that satisfies the "boundary conditions":

$$u(0,y) = u(\pi,y) = 0,\ u(x,0) = h(x),\ \lim_{y\to\infty} u(x,y) = 0.$$

(b) What would go awry if we omitted the condition at infinity?

Solution

A harmonic function $u(x,y)$ is a function that satisfies the Laplace equation, so the boundary value problem we have to solve is the following.

$$\begin{gathered}\nabla^2 u = 0, \quad 0 < x < \pi,\ 0 < y < \infty \\ u(x,0) = h(x), \quad u(0,y) = 0 \\ u(x,\infty) = 0, \quad u(\pi,y) = 0\end{gathered}$$

As all but one of the boundary conditions are homogeneous, the method of separation of variables can be applied to solve the PDE. Assume a product solution of the form $u = X(x)Y(y)$ and plug it into the PDE

$$u_{xx} + u_{yy} = 0 \quad \rightarrow \quad X''Y + XY'' = 0$$

and the homogeneous boundary conditions.

$$\begin{aligned} u(0,y) &= 0 & \rightarrow \quad X(0)Y(y) &= 0 & \rightarrow \quad X(0) &= 0 \\ u(\pi,y) &= 0 & \rightarrow \quad X(\pi)Y(y) &= 0 & \rightarrow \quad X(\pi) &= 0 \\ u(x,\infty) &= 0 & \rightarrow \quad X(x)Y(\infty) &= 0 & \rightarrow \quad Y(\infty) &= 0 \end{aligned}$$

Now separate variables in the PDE: bring all functions of x to the left side and all functions of y to the right side. Note that the final answer will be the same regardless of which side the minus sign is on.

$$X''Y + XY'' = 0 \quad \rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y}$$

The only way that a function of x can be equal to a function of y is if both are equal to a constant λ.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

Values of λ for which the boundary conditions are satisfied are called the eigenvalues, and the nontrivial functions associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

If λ is positive, then the ODE for X becomes

$$\frac{X''}{X} = \mu^2.$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Apply the boundary conditions here to determine C_1 and C_2.

$$X(0) = C_1 = 0$$
$$X(\pi) = C_1 \cosh \mu\pi + C_2 \sinh \mu\pi = 0$$

Since $C_1 = 0$, the second equation reduces to

$$C_2 \sinh \mu\pi = 0.$$

Because hyperbolic sine is not oscillatory, the equation is only satisfied if $C_2 = 0$. The trivial solution is obtained, so there are no position eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

If λ is zero, then the ODE for X becomes

$$\frac{X''}{X} = 0.$$

Multiply both sides by X.

$$X'' = 0$$

Integrate both sides with respect to x twice to solve for X.

$$X(x) = C_3 x + C_4$$

Apply the boundary conditions here to determine C_3 and C_4.

$$X(0) = C_4 = 0$$
$$X(\pi) = C_3\pi + C_4 = 0$$

Since $C_4 = 0$, the second equation is only satisfied if $C_3 = 0$. The trivial solution is obtained as a result, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

If λ is negative, then the ODE for X becomes

$$\frac{X''}{X} = -\gamma^2.$$

Multiply both sides by X.

$$X'' = -\gamma^2 X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_5 \cos \gamma x + C_6 \sin \gamma x$$

Apply the boundary conditions here to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_5 = 0 \\ X(\pi) &= C_5 \cos\gamma\pi + C_6 \sin\gamma\pi = 0 \end{aligned}$$

Since $C_5 = 0$, the second equation reduces to

$$C_6 \sin\gamma\pi = 0.$$

To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned} \sin\gamma\pi &= 0 \\ \gamma\pi &= n\pi, \quad n = 1, 2, \ldots \\ \gamma_n &= n, \quad n = 1, 2, \ldots. \end{aligned}$$

The eigenfunctions associated with these eigenvalues for λ are

$$X(x) = C_6 \sin\gamma x \quad \rightarrow \quad X_n(x) = \sin nx, \quad n = 1, 2, \ldots.$$

Because $X(x)$ is relevant, we now solve the related ODE for $Y(y)$.

$$-\frac{Y''}{Y} = -\gamma^2$$

Multiply both sides by $-Y$.

$$Y'' = \gamma^2 Y$$

The general solution can be written in terms of exponential functions.

$$Y(y) = C_7 e^{\gamma y} + C_8 e^{-\gamma y}$$

In order to satisfy the boundary condition at $y = \infty$, $Y(\infty) = 0$, we require $C_7 = 0$.

$$Y(y) = C_8 e^{-\gamma y} \quad \rightarrow \quad Y_n(y) = e^{-ny}, \quad n = 1, 2, \ldots$$

According to the principle of superposition, the general solution for u is a linear combination of the eigenfunctions $X(x)Y(y)$ over all the eigenvalues.

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} B_n e^{-ny} \sin nx}$$

To determine the coefficients B_n, we use the inhomogeneous boundary condition at $y = 0$.

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin nx = h(x)$$

Multiply both sides by $\sin mx$, where m is an integer.

$$\sum_{n=1}^{\infty} B_n \sin nx \sin mx = h(x) \sin mx$$

Integrate both sides with respect to x from 0 to π.

$$\int_0^\pi \sum_{n=1}^\infty B_n \sin nx \sin mx \, dx = \int_0^\pi h(x) \sin mx \, dx$$

Bring the constants in front of the integral on the left side.

$$\sum_{n=1}^\infty B_n \int_0^\pi \sin nx \sin mx \, dx = \int_0^\pi h(x) \sin mx \, dx$$

Because the sine functions are orthogonal, the integral on the left is zero for $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$B_n \int_0^\pi \sin^2 nx \, dx = \int_0^\pi h(x) \sin nx \, dx$$

Evaluate the integral on the left side.

$$B_n \cdot \frac{\pi}{2} = \int_0^\pi h(x) \sin nx \, dx$$

Therefore,

$$\boxed{B_n = \frac{2}{\pi} \int_0^\pi h(x) \sin nx \, dx.}$$

If the condition at infinity is omitted, then it is not the case that $C_7 = 0$, and the solution will diverge as $y \to \infty$.