Exercise 1

Suppose that u is a harmonic function in the disk $D = \{r < 2\}$ and that $u = 3\sin 2\theta + 1$ for $r = 2$. Without finding the solution, answer the following questions.

(a) Find the maximum value of u in $\overline{D}$.

(b) Calculate the value of u at the origin.

Solution

The fact that u is a harmonic function means that it satisfies Laplace's equation.

$$\nabla^2 u = 0$$

Part (a)

$\overline{D}$ is the union of D and its boundary: $\overline{D} = D \cup \operatorname{bdy} D = \{r \leq 2\}$. According to the maximum principle, the maximum value of u in $\overline{D}$ occurs on the boundary $r = 2$. Our task then is to find the maximum of the prescribed function $u(2, \theta) = 3\sin 2\theta + 1$.

$$\frac{d}{d\theta}(3\sin 2\theta + 1) = 6\cos 2\theta = 0 \quad \rightarrow \quad 2\theta = \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}\right\}$$

The critical values of θ are $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$.

$$\begin{aligned} u\left(2, \frac{\pi}{4}\right) &= 4 \\ u\left(2, \frac{3\pi}{4}\right) &= -2 \\ u\left(2, \frac{5\pi}{4}\right) &= 4 \\ u\left(2, \frac{7\pi}{4}\right) &= -2 \end{aligned}$$

Therefore, the maximum value of u in $\overline{D}$ is 4.

Part (b)

According to the mean value property, the value of u at the origin is the average of u on the circumference. What we have to do then is integrate the prescribed function $u(2, \theta)$ over the circumference and then divide the result by the circumference.

$$\begin{aligned} u(0) = \frac{\int u(2,\theta)\,ds}{\int ds} = \frac{\int_0^{2\pi} u(2,\theta)(2\,d\theta)}{\int_0^{2\pi} 2\,d\theta} = \frac{1}{2\pi}\int_0^{2\pi} u(2,\theta)\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi} (3\sin 2\theta + 1)\,d\theta \\ &= \frac{1}{2\pi}\left(-\frac{3}{2}\cos 2\theta + \theta\right)\Bigg|_0^{2\pi} \\ &= 1 \end{aligned}$$

Therefore, the value of u at the origin is 1.