Exercise 3

Same for the boundary condition $u = \sin^3\theta$. (*Hint:* Use the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.)

Solution

The PDE to solve is Laplace's equation,

$$\nabla^2 u = 0,$$

subject to the boundary condition,

$$\begin{aligned} u(a,\theta) &= \sin^3\theta \\ &= \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta. \end{aligned}$$

Since the boundary condition is given on a circle of radius a, we opt to write the Laplacian operator ∇^2 in polar coordinates and solve for u as a function of r and θ.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \tag{1}$$

From the form of the boundary condition we hypothesize that the solution has the form

$$u(r,\theta) = f(r)\sin\theta + g(r)\sin 3\theta.$$

In order to determine $f(r)$ and $g(r)$, substitute this expression for $u(r,\theta)$ into equation (1).

$$\frac{\partial^2}{\partial r^2}[f(r)\sin\theta + g(r)\sin 3\theta] + \frac{1}{r}\frac{\partial}{\partial r}[f(r)\sin\theta + g(r)\sin 3\theta] + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}[f(r)\sin\theta + g(r)\sin 3\theta] = 0$$

Evaluate the derivatives.

$$f''(r)\sin\theta + g''(r)\sin 3\theta + \frac{1}{r}[f'(r)\sin\theta + g'(r)\sin 3\theta] + \frac{1}{r^2}[f(r)(-\sin\theta) + g(r)(-9\sin 3\theta)] = 0$$

Expand the left side.

$$f''(r)\sin\theta + \frac{1}{r}f'(r)\sin\theta - \frac{1}{r^2}f(r)\sin\theta + g''(r)\sin 3\theta + \frac{1}{r}g'(r)\sin 3\theta - \frac{9}{r^2}g(r)\sin 3\theta = 0$$

If we set

$$f''(r)\sin\theta + \frac{1}{r}f'(r)\sin\theta - \frac{1}{r^2}f(r)\sin\theta = 0,$$

then the previous equation reduces to

$$g''(r)\sin 3\theta + \frac{1}{r}g'(r)\sin 3\theta - \frac{9}{r^2}g(r)\sin 3\theta = 0.$$

Multiply both sides of the ODE for $f(r)$ by $r^2/\sin\theta$, and multiply both sides of the ODE for $g(r)$ by $r^2/\sin 3\theta$.

$$\begin{aligned} r^2 f'' + rf' - f &= 0 \\ r^2 g'' + rg' - 9g &= 0 \end{aligned}$$

Since θ is not present in either equation, the hypothesis for $u(r,\theta)$ is legitimate. Both are equidimensional ODEs for f and g, so they have solutions of the form

$$\begin{aligned} f(r) &= r^m &\rightarrow\quad f'(r) &= mr^{m-1} &\rightarrow\quad f''(r) &= m(m-1)r^{m-2} \\ g(r) &= r^n &\rightarrow\quad g'(r) &= nr^{n-1} &\rightarrow\quad g''(r) &= n(n-1)r^{n-2}. \end{aligned}$$

Substitute these expressions into the respective ODEs to determine the constants, m and n.

$$\begin{aligned} m(m-1)r^m + mr^m - r^m &= 0 & n(n-1)r^n + nr^n - 9r^n &= 0 \\ m(m-1) + m - 1 &= 0 & n(n-1) + n - 9 &= 0 \\ m^2 - 1 &= 0 & n^2 - 9 &= 0 \\ m &= \{\pm 1\} & n &= \{\pm 3\} \end{aligned}$$

Consequently,

$$\begin{aligned} f(r) &= C_1 r + C_2 r^{-1} \\ g(r) &= C_3 r^3 + C_4 r^{-3}. \end{aligned}$$

The constants are determined by using the boundary conditions, $f(a) = 3/4$, $f(0) =$ finite, $g(a) = -1/4$, and $g(0) =$ finite. To satisfy the conditions at $r = 0$, we require $C_2 = 0$ and $C_4 = 0$.

$$\begin{aligned} f(a) &= C_1 a = \frac{3}{4} &\rightarrow\quad C_1 &= \frac{3}{4a} \\ g(a) &= C_3 a^3 = -\frac{1}{4} &\rightarrow\quad C_3 &= -\frac{1}{4a^3} \end{aligned}$$

So then

$$\begin{aligned} f(r) &= \frac{3r}{4a} \\ g(r) &= -\frac{r^3}{4a^3}. \end{aligned}$$

Therefore,

$$u(r,\theta) = \frac{3r}{4a}\sin\theta - \frac{r^3}{4a^3}\sin 3\theta.$$

Setting $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$ gives us the function in Cartesian coordinates and allows us to plot it.

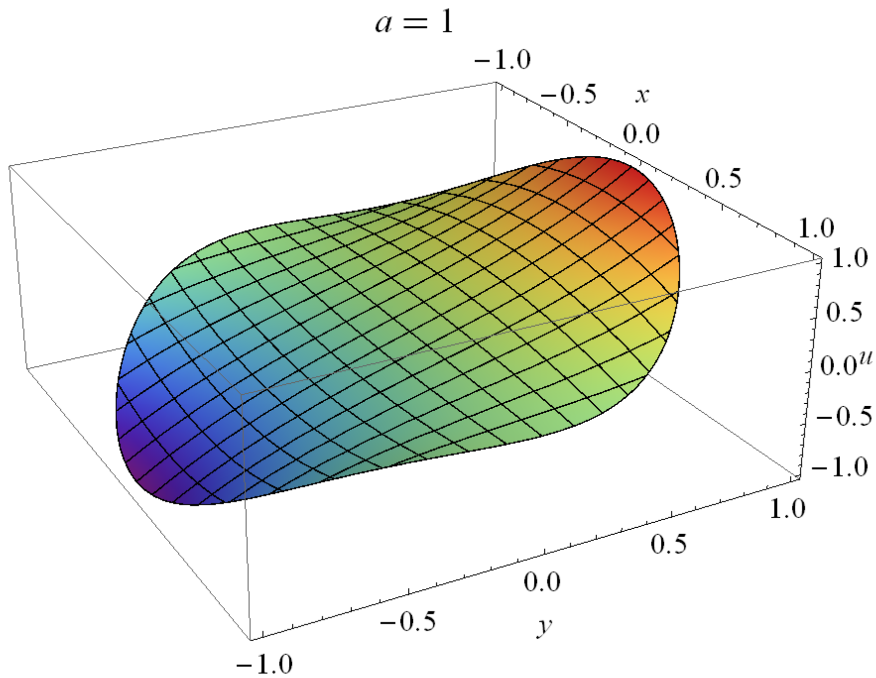

Figure 1: This is a plot of the two-dimensional solution surface $u(x, y)$ in three-dimensional xyu-space for $a = 1$. Notice that the maximum and minimum values of u lie on the boundary (maximum principle) and that the value of u at the origin is the average of values along the boundary (mean value property).