Exercise 4

Show that $P(r, \theta)$ is a harmonic function in D by using polar coordinates. That is, use (6.1.5) on the first expression in (17).

Solution

The aim here is to show that the Poisson kernel $P(r, \theta)$ satisfies the Laplace equation in polar coordinates, that is,

$$P_{rr} + \frac{1}{r}P_r + \frac{1}{r^2}P_{\theta\theta} = 0.$$

The Poisson kernel is defined as

$$P(r, \theta) = \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2},$$

so the derivatives are as follows.

$$\begin{aligned}
P_r &= \frac{(-2r)(a^2 - 2ar\cos\theta + r^2) - (a^2 - r^2)(2r - 2a\cos\theta)}{(a^2 - 2ar\cos\theta + r^2)^2} \\
&= \frac{2ar^2\cos\theta + 2a^3\cos\theta - 4a^2r}{(a^2 - 2ar\cos\theta + r^2)^2} \cdot \frac{a^2 - 2ar\cos\theta + r^2}{a^2 - 2ar\cos\theta + r^2} \\
&= \frac{-4a^4r - 4a^2r^3 + 2a^5\cos\theta + 12a^3r^2\cos\theta + 2ar^4\cos\theta - 4a^4r\cos^2\theta - 4a^2r^3\cos^2\theta}{(a^2 - 2ar\cos\theta + r^2)^3} \\
P_{rr} &= \frac{(4ar\cos\theta - 4a^2)(a^2 - 2ar\cos\theta + r^2)^2 - 2(a^2 - 2ar\cos\theta + r^2)(-2a\cos\theta + 2r)(2ar^2\cos\theta + 2a^3\cos\theta - 4a^2r)}{(a^2 - 2ar\cos\theta + r^2)^4} \\
&= \frac{(4ar\cos\theta - 4a^2)(a^2 - 2ar\cos\theta + r^2) - 2(-2a\cos\theta + 2r)(2ar^2\cos\theta + 2a^3\cos\theta - 4a^2r)}{(a^2 - 2ar\cos\theta + r^2)^3} \\
&= \frac{8a^4\cos^2\theta - 4ar^3\cos\theta - 12a^3r\cos\theta + 12a^2r^2 - 4a^4}{(a^2 - 2ar\cos\theta + r^2)^3} \\
P_\theta &= \frac{0 - (2ar\sin\theta)(a^2 - r^2)}{(a^2 - 2ar\cos\theta + r^2)^2} \\
&= \frac{2ar\sin\theta(r^2 - a^2)}{(a^2 - 2ar\cos\theta + r^2)^2} \\
P_{\theta\theta} &= \frac{2ar\cos\theta(r^2 - a^2)(a^2 - 2ar\cos\theta + r^2)^2 - 2(a^2 - 2ar\cos\theta + r^2)(2ar\sin\theta)[2ar\sin\theta(r^2 - a^2)]}{(a^2 - 2ar\cos\theta + r^2)^4} \\
&= \frac{2ar\cos\theta(r^2 - a^2)(a^2 - 2ar\cos\theta + r^2) - 2(2ar\sin\theta)[2ar\sin\theta(r^2 - a^2)]}{(a^2 - 2ar\cos\theta + r^2)^3} \\
&= \frac{8a^4r^2\sin^2\theta - 8a^2r^4\sin^2\theta + 4a^4r^2\cos^2\theta - 4a^2r^4\cos^2\theta + 2ar^5\cos\theta - 2a^5r\cos\theta}{(a^2 - 2ar\cos\theta + r^2)^3}
\end{aligned}$$

So then

$$\begin{aligned}
P_{rr} + \frac{1}{r}P_r + \frac{1}{r^2}P_{\theta\theta} &= \\
&\frac{8a^4\cos^2\theta - \cancel{4ar^3\cos\theta} - 12a^3 r\cos\theta + 12a^2r^2 - 4a^4}{(a^2 - 2ar\cos\theta + r^2)^3} \\
&+ \frac{-4a^4 - 4a^2r^2 + \cancel{\frac{2a^5\cos\theta}{r}} + 12a^3r\cos\theta + \cancel{2ar^3\cos\theta} - \cancel{4a^4\cos^2\theta} - 4a^2r^2\cos^2\theta}{(a^2 - 2ar\cos\theta + r^2)^3} \\
&+ \frac{8a^4\sin^2\theta - 8a^2r^2\sin^2\theta + \cancel{4a^4\cos^2\theta} - 4a^2r^2\cos^2\theta + \cancel{2ar^3\cos\theta} - \cancel{\frac{2a^5\cos\theta}{r}}}{(a^2 - 2ar\cos\theta + r^2)^3} \stackrel{?}{=} 0
\end{aligned}$$

Multiply both sides by $(a^2 - 2ar\cos\theta + r^2)^3$.

$$\begin{aligned}
\cancel{8a^4\cos^2\theta} - \cancel{12a^3r\cos\theta} + 12a^2r^2 - \cancel{4a^4} - \cancel{4a^4} - 4a^2r^2 + \cancel{12a^3r\cos\theta} - 4a^2r^2\cos^2\theta \\
+ \cancel{8a^4\sin^2\theta} - 8a^2r^2\sin^2\theta - 4a^2r^2\cos^2\theta \stackrel{?}{=} 0
\end{aligned}$$

All that remains is

$$\begin{aligned}
8a^2r^2 - 8a^2r^2\sin^2\theta - 8a^2r^2\cos^2\theta &\stackrel{?}{=} 0 \\
8a^2r^2 - 8a^2r^2 &\stackrel{?}{=} 0 \\
0 &= 0.
\end{aligned}$$

$P(r, \theta)$ satisfies the Laplace equation; therefore, it is a harmonic function in D.