

Exercise 1

Solve $u_{xx} + u_{yy} = 0$ in the *exterior* $\{r > a\}$ of a disk, with the boundary condition $u = 1 + 3 \sin \theta$ on $r = a$, and the condition at infinity that u be bounded as $r \rightarrow \infty$.

Solution

The PDE to solve is Laplace's equation.

$$\nabla^2 u = 0$$

Since the boundary condition is given on a circle of radius a , we opt to write the Laplacian operator ∇^2 in polar coordinates and solve for u as a function of r and θ .

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad (1)$$

From the form of the boundary condition we hypothesize that the solution has the form

$$u(r, \theta) = 1 + f(r) \sin \theta.$$

In order to determine $f(r)$, substitute this expression for $u(r, \theta)$ into equation (1).

$$\frac{\partial^2}{\partial r^2}[1 + f(r) \sin \theta] + \frac{1}{r} \frac{\partial}{\partial r}[1 + f(r) \sin \theta] + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}[1 + f(r) \sin \theta] = 0$$

Evaluate the derivatives.

$$f''(r) \sin \theta + \frac{1}{r} f'(r) \sin \theta + \frac{1}{r^2} f(r) (-\sin \theta) = 0$$

Multiply both sides by $r^2 / \sin \theta$.

$$r^2 f'' + r f' - f = 0$$

Since θ is not present in this equation, the hypothesis for $u(r, \theta)$ is legitimate. This is an equidimensional ODE for f , so it has solutions of the form

$$f(r) = r^m \quad \rightarrow \quad f'(r) = m r^{m-1} \quad \rightarrow \quad f''(r) = m(m-1) r^{m-2}.$$

Substitute these expressions into the ODE to determine the constants m .

$$m(m-1)r^m + m r^m - r^m = 0$$

Divide both sides by r^m .

$$m(m-1) + m - 1 = 0$$

Solve for m .

$$m^2 - 1 = 0 \quad \rightarrow \quad m = \{\pm 1\}$$

Consequently,

$$f(r) = C_1 r + C_2 r^{-1}.$$

C_1 and C_2 are determined by using the boundary conditions, $f(a) = 3$ and $f(\infty) = \text{finite}$.

$$\begin{aligned} f(a) &= C_1 a + \frac{C_2}{a} = 3 \\ f(\infty) &= \text{bounded} \quad \rightarrow \quad C_1 = 0 \end{aligned}$$

Because $C_1 = 0$, the first equation reduces to $C_2/a = 3$, so $C_2 = 3a$.

$$f(r) = \frac{3a}{r}$$

The solution for u in polar coordinates is then

$$u(r, \theta) = 1 + \frac{3a}{r} \sin \theta.$$

Since $y = r \sin \theta$, the solution in Cartesian coordinates is therefore

$$u(x, y) = 1 + \frac{3a}{x^2 + y^2} y.$$

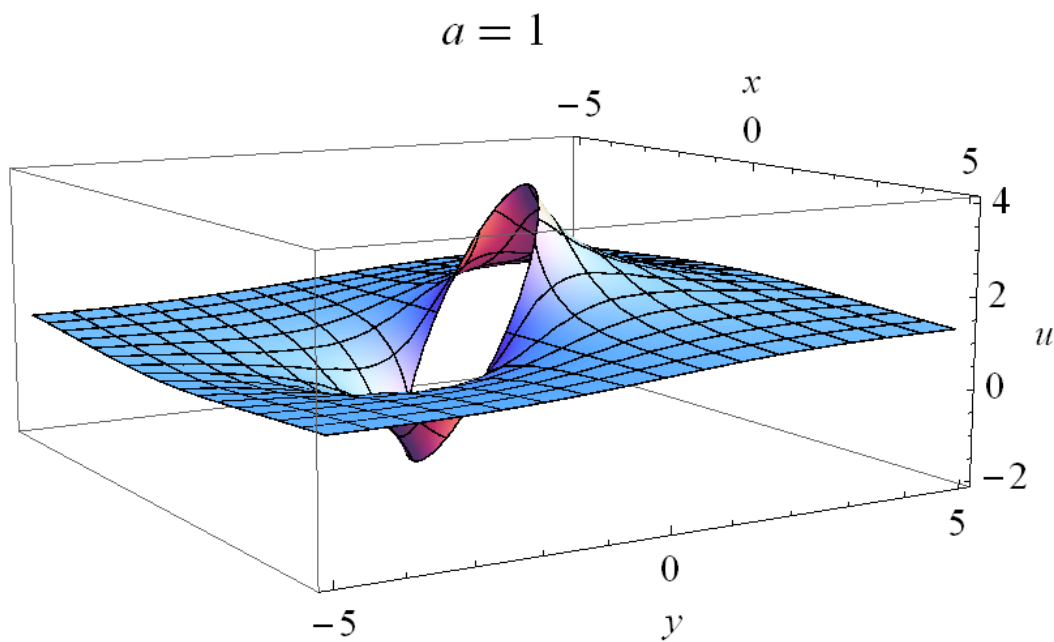


Figure 1: This is a plot of the two-dimensional solution surface $u(x, y)$ in three-dimensional xyu -space for $a = 1$.