Exercise 13

Solve $u_{xx} + u_{yy} = 0$ in the region $\{\alpha < \theta < \beta,\ a < r < b\}$ with the boundary conditions $u = 0$ on the two sides $\theta = \alpha$ and $\theta = \beta$, $u = g(\theta)$ on the arc $r = a$, and $u = h(\theta)$ on the arc $r = b$.

Solution

Since the boundary conditions are given along certain angles and radii, we choose to expand the Laplacian operator in polar coordinates. The boundary value problem to solve then is the following.

$$\begin{gathered} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad a < r < b,\ \alpha < \theta < \beta \\ u(a,\theta) = g(\theta), \quad u(r,\alpha) = 0 \\ u(b,\theta) = h(\theta), \quad u(r,\beta) = 0 \end{gathered}$$

The method of separation of variables cannot be applied at the moment because there are two inhomogeneous boundary conditions. However, we can take advantage of the fact that the Laplace equation is linear and split the problem up into two simpler ones that can be solved with the method. Let $u = v + w$, where v and w satisfy similar problems with only one inhomogeneous boundary condition in each.

$$\begin{aligned} v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} &= 0 & \qquad w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta} &= 0 \\ v(a,\theta) = g(\theta), \quad v(r,\alpha) &= 0 & w(a,\theta) = 0, \quad w(r,\alpha) &= 0 \\ v(b,\theta) = 0, \quad v(r,\beta) &= 0 & w(b,\theta) = h(\theta), \quad w(r,\beta) &= 0 \end{aligned}$$

Now that all but one boundary condition are homogeneous in each problem, the method of separation of variables can be applied to solve each one. Assume product solutions of the forms, $v = R_1(r)\Theta_1(\theta)$ and $w = R_2(r)\Theta_2(\theta)$ and plug them into the PDEs

$$R_1''\Theta_1 + \frac{1}{r}R_1'\Theta_1 + \frac{1}{r^2}R_1\Theta_1'' = 0 \qquad R_2''\Theta_2 + \frac{1}{r}R_2'\Theta_2 + \frac{1}{r^2}R_2\Theta_2'' = 0$$

and the homogeneous boundary conditions.

$$\begin{aligned} v(r,\alpha) = 0 \quad &\rightarrow \quad \Theta_1(\alpha) = 0 & \qquad w(r,\alpha) = 0 \quad &\rightarrow \quad \Theta_2(\alpha) = 0 \\ v(r,\beta) = 0 \quad &\rightarrow \quad \Theta_1(\beta) = 0 & w(r,\beta) = 0 \quad &\rightarrow \quad \Theta_2(\beta) = 0 \\ v(b,\theta) = 0 \quad &\rightarrow \quad R_1(b) = 0 & w(a,\theta) = 0 \quad &\rightarrow \quad R_2(a) = 0 \end{aligned}$$

Now separate variables in the PDEs: bring the functions of r to the left sides and the functions of θ to the right sides. Note that the final answers will be the same regardless of which sides the minus signs are on.

$$r^2\frac{R_1''}{R_1} + r\frac{R_1'}{R_1} = -\frac{\Theta_1''}{\Theta_1} \qquad r^2\frac{R_2''}{R_2} + r\frac{R_2'}{R_2} = -\frac{\Theta_2''}{\Theta_2}$$

The only way that a function of r can be equal to a function of θ is if both are equal to a constant. For the two equations, the constants are not necessarily the same.

$$r^2\frac{R_1''}{R_1} + r\frac{R_1'}{R_1} = -\frac{\Theta_1''}{\Theta_1} = \lambda_1 \qquad r^2\frac{R_2''}{R_2} + r\frac{R_2'}{R_2} = -\frac{\Theta_2''}{\Theta_2} = \lambda_2$$

Values of λ_1 and λ_2 for which the boundary conditions are satisfied are called the eigenvalues, and the nontrivial functions associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda_1 = \mu^2$ and $\lambda_2 = \xi^2$

If λ_1 and λ_2 are positive, then the ODEs for Θ_1 and Θ_2 become

$$\begin{aligned} -\frac{\Theta_1''}{\Theta_1} &= \mu^2 & -\frac{\Theta_2''}{\Theta_2} &= \xi^2 \\ \Theta_1'' &= -\mu^2\Theta_1 & \Theta_2'' &= -\xi^2\Theta_2 \\ \Theta_1(\theta) &= C_1\cos\mu\theta + C_2\sin\mu\theta & \Theta_2(\theta) &= C_3\cos\xi\theta + C_4\sin\xi\theta. \end{aligned}$$

Apply the boundary conditions now to determine the constants.

$$\begin{aligned} \Theta_1(\alpha) &= C_1\cos\mu\alpha + C_2\sin\mu\alpha = 0 & \Theta_2(\alpha) &= C_3\cos\xi\alpha + C_4\sin\xi\alpha = 0 \\ \Theta_1(\beta) &= C_1\cos\mu\beta + C_2\sin\mu\beta = 0 & \Theta_2(\beta) &= C_3\cos\xi\beta + C_4\sin\xi\beta = 0 \end{aligned}$$

Solve $\Theta_1(\alpha) = 0$ for C_1 and solve $\Theta_2(\alpha) = 0$ for C_3.

$$C_1 = -C_2\frac{\sin\mu\alpha}{\cos\mu\alpha} \qquad C_3 = -C_4\frac{\sin\xi\alpha}{\cos\xi\alpha}$$

Substitute these results into $\Theta_1(\beta) = 0$ and $\Theta_2(\beta) = 0$, respectively.

$$-C_2\frac{\sin\mu\alpha}{\cos\mu\alpha}\cos\mu\beta + C_2\sin\mu\beta = 0 \qquad -C_4\frac{\sin\xi\alpha}{\cos\xi\alpha}\cos\xi\beta + C_4\sin\xi\beta = 0$$

To avoid getting the trivial solution, we insist that $C_2 \neq 0$ and $C_4 \neq 0$. Multiply both sides of the equation on the left by $(\cos\mu\alpha)/C_2$ and both sides of the equation on the right by $(\cos\xi\alpha)/C_4$.

$$\begin{aligned} \sin\mu\beta\cos\mu\alpha - \cos\mu\beta\sin\mu\alpha &= 0 & \sin\xi\beta\cos\xi\alpha - \cos\xi\beta\sin\xi\alpha &= 0 \\ \sin[\mu(\beta-\alpha)] &= 0 & \sin[\xi(\beta-\alpha)] &= 0 \\ \mu(\beta-\alpha) &= n\pi & \xi(\beta-\alpha) &= n\pi \\ \mu_n &= \frac{n\pi}{\beta-\alpha} & \xi_n &= \frac{n\pi}{\beta-\alpha}, \end{aligned}$$

where $n = 1, 2, \ldots$. The eigenfunctions associated with these eigenvalues for λ_1 and λ_2 are

$$\begin{aligned} \Theta_1(\theta) &= C_1\cos\mu\theta + C_2\sin\mu\theta & \Theta_2(\theta) &= C_3\cos\xi\theta + C_4\sin\xi\theta \\ \Theta_1(\theta) &= -C_2\frac{\sin\mu\alpha}{\cos\mu\alpha}\cos\mu\theta + C_2\sin\mu\theta & \Theta_2(\theta) &= -C_4\frac{\sin\xi\alpha}{\cos\xi\alpha}\cos\xi\theta + C_4\sin\xi\theta \\ \Theta_1(\theta) &= C_2\frac{\sin\mu\theta\cos\mu\alpha - \cos\mu\theta\sin\mu\alpha}{\cos\mu\alpha} & \Theta_2(\theta) &= C_4\frac{\sin\xi\theta\cos\xi\alpha - \cos\xi\theta\sin\xi\alpha}{\cos\xi\alpha} \\ \Theta_1(\theta) &= C_2\frac{\sin[\mu(\theta-\alpha)]}{\cos\mu\alpha} & \Theta_2(\theta) &= C_4\frac{\sin[\xi(\theta-\alpha)]}{\cos\xi\alpha} \\ \Theta_{1n}(\theta) &= \sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right] & \Theta_{2n}(\theta) &= \sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right], \end{aligned}$$

where $n = 1, 2, \ldots$. Now the related ODEs for R_1 and R_2 will be solved.

$$\begin{aligned} r^2\frac{R_1''}{R_1} + r\frac{R_1'}{R_1} &= \mu^2 & r^2\frac{R_2''}{R_2} + r\frac{R_2'}{R_2} &= \xi^2 \\ r^2R_1'' + rR_1' - \mu^2R_1 &= 0 & r^2R_2'' + rR_2' - \xi^2R_2 &= 0 \end{aligned}$$

Since μ and ξ are equal, R_1 and R_2 will have the same general solution. The ODEs are equidimensional, so the solutions are of the form, $R_1(r) = r^m$ and $R_2(r) = r^m$. Plug them into the ODEs to find m.

$$\begin{aligned} m(m-1)r^m + mr^m - \mu^2 r^m &= 0 \\ m^2 - \mu^2 &= 0 \\ m &= \{\pm\mu\} \end{aligned} \qquad \begin{aligned} m(m-1)r^m + mr^m - \xi^2 r^m &= 0 \\ m^2 - \xi^2 &= 0 \\ m &= \{\pm\xi\} \end{aligned}$$

So then

$$R_1(r) = C_5 r^{\mu} + C_6 r^{-\mu} \qquad R_2(r) = C_7 r^{\xi} + C_8 r^{-\xi}.$$

Apply the boundary conditions at $r = b$ and $r = a$ to determine two of the constants.

$$R_1(b) = C_5 b^{\mu} + C_6 b^{-\mu} = 0 \qquad R_2(a) = C_7 a^{\xi} + C_8 a^{-\xi} = 0$$

Solve $R_1(b) = 0$ for C_6 and solve $R_2(a) = 0$ for C_8.

$$C_6 = -C_5 b^{2\mu} \qquad C_8 = -C_7 a^{2\xi}$$

Consequently,

$$\begin{aligned} R_1(r) &= C_5 r^{\mu} - C_5 b^{2\mu} r^{-\mu} \\ R_1(r) &= C_5 r^{\mu}\left(1 - \frac{b^{2\mu}}{r^{2\mu}}\right) \\ R_1(r) &= C_5 r^{\mu}\left[1 - \left(\frac{b}{r}\right)^{2\mu}\right] \\ R_{1n}(r) &= r^{n\pi/(\beta-\alpha)}\left[1 - \left(\frac{b}{r}\right)^{2n\pi/(\beta-\alpha)}\right] \end{aligned} \qquad \begin{aligned} R_2(r) &= C_7 r^{\xi} - C_7 a^{2\xi} r^{-\xi} \\ R_2(r) &= C_7 r^{\xi}\left(1 - \frac{a^{2\xi}}{r^{2\xi}}\right) \\ R_2(r) &= C_7 r^{\xi}\left[1 - \left(\frac{a}{r}\right)^{2\xi}\right] \\ R_{2n}(r) &= r^{n\pi/(\beta-\alpha)}\left[1 - \left(\frac{a}{r}\right)^{2n\pi/(\beta-\alpha)}\right], \end{aligned}$$

where $n = 1, 2, \ldots$.

Determination of the Zero Eigenvalue: $\lambda_1 = 0$ and $\lambda_2 = 0$

If λ_1 and λ_2 are zero, then the ODEs for Θ_1 and Θ_2 become

$$\begin{aligned} -\frac{\Theta_1''}{\Theta_1} &= 0 \\ \Theta_1'' &= 0 \\ \Theta_1(\theta) &= C_9\theta + C_{10} \end{aligned} \qquad \begin{aligned} -\frac{\Theta_2''}{\Theta_2} &= 0 \\ \Theta_2'' &= 0 \\ \Theta_2(\theta) &= C_{11}\theta + C_{12} \end{aligned}$$

Apply the boundary conditions to determine the constants.

$$\begin{aligned} \Theta_1(\alpha) &= C_9\alpha + C_{10} = 0 \\ \Theta_1(\beta) &= C_9\beta + C_{10} = 0 \end{aligned} \qquad \begin{aligned} \Theta_2(\alpha) &= C_{11}\alpha + C_{12} = 0 \\ \Theta_2(\beta) &= C_{11}\beta + C_{12} = 0 \end{aligned}$$

Solving the two systems yields $C_9 = 0$, $C_{10} = 0$, $C_{11} = 0$, and $C_{12} = 0$. The trivial solution is obtained in both cases, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda_1 = -\gamma^2$ and $\lambda_2 = -\zeta^2$

If λ_1 and λ_2 are negative, then the ODEs for Θ_1 and Θ_2 become

$$-\frac{\Theta_1''}{\Theta_1} = -\gamma^2 \qquad\qquad -\frac{\Theta_2''}{\Theta_2} = -\zeta^2$$
$$\Theta_1'' = \gamma^2\Theta_1 \qquad\qquad \Theta_2'' = \zeta^2\Theta_2$$
$$\Theta_1(\theta) = C_{13}\cosh\gamma\theta + C_{14}\sinh\gamma\theta \qquad\qquad \Theta_2(\theta) = C_{15}\cosh\zeta\theta + C_{16}\sinh\zeta\theta.$$

Apply the boundary conditions now to determine the constants.

$$\Theta_1(\alpha) = C_{13}\cosh\gamma\alpha + C_{14}\sinh\gamma\alpha = 0 \qquad \Theta_2(\alpha) = C_{15}\cosh\zeta\alpha + C_{16}\sinh\zeta\alpha = 0$$
$$\Theta_1(\beta) = C_{13}\cosh\gamma\beta + C_{14}\sinh\gamma\beta = 0 \qquad \Theta_2(\beta) = C_{15}\cosh\zeta\beta + C_{16}\sinh\zeta\beta = 0$$

Solve $\Theta_1(\alpha) = 0$ for C_{13} and solve $\Theta_2(\alpha) = 0$ for C_{15}.

$$C_{13} = -C_{14}\frac{\sinh\gamma\alpha}{\cosh\gamma\alpha} \qquad\qquad C_{15} = -C_{16}\frac{\sinh\zeta\alpha}{\cosh\zeta\alpha}$$

Substitute these results into $\Theta_1(\beta) = 0$ and $\Theta_2(\beta) = 0$, respectively.

$$-C_{14}\frac{\sinh\gamma\alpha}{\cosh\gamma\alpha}\cosh\gamma\beta + C_{14}\sinh\gamma\beta = 0 \qquad -C_{16}\frac{\sinh\zeta\alpha}{\cosh\zeta\alpha}\cosh\zeta\beta + C_{16}\sinh\zeta\beta = 0$$

Multiply both sides of the equation on the left by $\cosh\gamma\alpha$ and both sides of the equation on the right by $\cosh\zeta\alpha$.

$$C_{14}(\sinh\gamma\beta\cosh\gamma\alpha - \cosh\gamma\beta\sinh\gamma\alpha) = 0 \qquad C_{16}(\sinh\zeta\beta\cosh\zeta\alpha - \cosh\zeta\beta\sinh\zeta\alpha) = 0$$
$$C_{14}\sinh[\gamma(\beta-\alpha)] = 0 \qquad\qquad C_{16}\sinh[\zeta(\beta-\alpha)] = 0$$

Hyperbolic sine is not oscillatory, so the only way these equations are satisfied is if $C_{14} = 0$ and $C_{16} = 0$. It follows that $C_{13} = 0$ and $C_{15} = 0$ as well. The trivial solution is obtained in both cases, so there are no negative eigenvalues.

According to the principle of superposition, the general solutions for v and w are linear combinations of the eigenfunctions $R_1(r)\Theta_1(\theta)$ and $R_2(r)\Theta_2(\theta)$, respectively, over all the eigenvalues.

$$v(r,\theta) = \sum_{n=1}^{\infty} A_n r^{n\pi/(\beta-\alpha)}\left[1 - \left(\frac{b}{r}\right)^{2n\pi/(\beta-\alpha)}\right]\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]$$
$$w(r,\theta) = \sum_{n=1}^{\infty} B_n r^{n\pi/(\beta-\alpha)}\left[1 - \left(\frac{a}{r}\right)^{2n\pi/(\beta-\alpha)}\right]\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]$$

The aim now is to determine the coefficients, A_n and B_n, by using the inhomogeneous boundary conditions. Start by using $v(a,\theta) = g(\theta)$ to find A_n.

$$v(a,\theta) = \sum_{n=1}^{\infty} A_n a^{n\pi/(\beta-\alpha)}\left[1 - \left(\frac{b}{a}\right)^{2n\pi/(\beta-\alpha)}\right]\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right] = g(\theta)$$

Multiply both sides by $\sin\left[\frac{p\pi}{\beta-\alpha}(\theta-\alpha)\right]$, where p is an integer.

$$\sum_{n=1}^{\infty} A_n a^{n\pi/(\beta-\alpha)}\left[1-\left(\frac{b}{a}\right)^{2n\pi/(\beta-\alpha)}\right]\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]\sin\left[\frac{p\pi}{\beta-\alpha}(\theta-\alpha)\right]=g(\theta)\sin\left[\frac{p\pi}{\beta-\alpha}(\theta-\alpha)\right]$$

Integrate both sides with respect to θ from α to β.

$$\int_{\alpha}^{\beta}\sum_{n=1}^{\infty} A_n a^{n\pi/(\beta-\alpha)}\left[1-\left(\frac{b}{a}\right)^{2n\pi/(\beta-\alpha)}\right]\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]\sin\left[\frac{p\pi}{\beta-\alpha}(\theta-\alpha)\right]d\theta$$
$$=\int_{\alpha}^{\beta} g(\theta)\sin\left[\frac{p\pi}{\beta-\alpha}(\theta-\alpha)\right]d\theta$$

Bring the constants in front of the integral on the left side.

$$\sum_{n=1}^{\infty} A_n a^{n\pi/(\beta-\alpha)}\left[1-\left(\frac{b}{a}\right)^{2n\pi/(\beta-\alpha)}\right]\int_{\alpha}^{\beta}\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]\sin\left[\frac{p\pi}{\beta-\alpha}(\theta-\alpha)\right]d\theta$$
$$=\int_{\alpha}^{\beta} g(\theta)\sin\left[\frac{p\pi}{\beta-\alpha}(\theta-\alpha)\right]d\theta$$

Since the sine functions are orthogonal, the integral is equal to zero for $n \neq p$. As a result, every term in the infinite series vanishes except for one: $n = p$.

$$A_n a^{n\pi/(\beta-\alpha)}\left[1-\left(\frac{b}{a}\right)^{2n\pi/(\beta-\alpha)}\right]\int_{\alpha}^{\beta}\sin^2\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]d\theta=\int_{\alpha}^{\beta} g(\theta)\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]d\theta$$

Evaluate the integral on the left side.

$$A_n a^{n\pi/(\beta-\alpha)}\left[1-\left(\frac{b}{a}\right)^{2n\pi/(\beta-\alpha)}\right]\cdot\frac{\beta-\alpha}{2}=\int_{\alpha}^{\beta} g(\theta)\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]d\theta$$

Solve this equation for A_n.

$$\boxed{A_n=\frac{2}{(\beta-\alpha)a^{n\pi/(\beta-\alpha)}\left[1-\left(\frac{b}{a}\right)^{2n\pi/(\beta-\alpha)}\right]}\int_{\alpha}^{\beta} g(\theta)\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]d\theta}$$

Now use $w(b,\theta)=h(\theta)$ to find B_n.

$$w(b,\theta)=\sum_{n=1}^{\infty} B_n b^{n\pi/(\beta-\alpha)}\left[1-\left(\frac{a}{b}\right)^{2n\pi/(\beta-\alpha)}\right]\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]=h(\theta)$$

Multiply both sides by $\sin\left[\frac{p\pi}{\beta-\alpha}(\theta-\alpha)\right]$, where p is an integer.

$$\sum_{n=1}^{\infty} B_n b^{n\pi/(\beta-\alpha)}\left[1-\left(\frac{a}{b}\right)^{2n\pi/(\beta-\alpha)}\right]\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]\sin\left[\frac{p\pi}{\beta-\alpha}(\theta-\alpha)\right]=h(\theta)\sin\left[\frac{p\pi}{\beta-\alpha}(\theta-\alpha)\right]$$

Integrate both sides with respect to θ from α to β.

$$\int_\alpha^\beta \sum_{n=1}^\infty B_n b^{n\pi/(\beta-\alpha)}\left[1-\left(\frac{a}{b}\right)^{2n\pi/(\beta-\alpha)}\right]\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]\sin\left[\frac{p\pi}{\beta-\alpha}(\theta-\alpha)\right]d\theta$$
$$= \int_\alpha^\beta h(\theta)\sin\left[\frac{p\pi}{\beta-\alpha}(\theta-\alpha)\right]d\theta$$

Bring the constants in front of the integral on the left side.

$$\sum_{n=1}^\infty B_n b^{n\pi/(\beta-\alpha)}\left[1-\left(\frac{a}{b}\right)^{2n\pi/(\beta-\alpha)}\right]\int_\alpha^\beta \sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]\sin\left[\frac{p\pi}{\beta-\alpha}(\theta-\alpha)\right]d\theta$$
$$= \int_\alpha^\beta h(\theta)\sin\left[\frac{p\pi}{\beta-\alpha}(\theta-\alpha)\right]d\theta$$

Since the sine functions are orthogonal, the integral is equal to zero for $n \neq p$. As a result, every term in the infinite series vanishes except for one: $n = p$.

$$B_n b^{n\pi/(\beta-\alpha)}\left[1-\left(\frac{a}{b}\right)^{2n\pi/(\beta-\alpha)}\right]\int_\alpha^\beta \sin^2\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]d\theta = \int_\alpha^\beta h(\theta)\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]d\theta$$

Evaluate the integral on the left side.

$$B_n b^{n\pi/(\beta-\alpha)}\left[1-\left(\frac{a}{b}\right)^{2n\pi/(\beta-\alpha)}\right]\cdot\frac{\beta-\alpha}{2} = \int_\alpha^\beta h(\theta)\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]d\theta$$

Solve this equation for B_n.

$$\boxed{B_n = \frac{2}{(\beta-\alpha)b^{n\pi/(\beta-\alpha)}\left[1-\left(\frac{a}{b}\right)^{2n\pi/(\beta-\alpha)}\right]}\int_\alpha^\beta h(\theta)\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right]d\theta}$$

The solution for u is obtained by adding v and w together.

$$\begin{aligned} u(r,\theta) &= v(r,\theta) + w(r,\theta) \\ &= \sum_{n=1}^\infty A_n r^{n\pi/(\beta-\alpha)}\left[1-\left(\frac{b}{r}\right)^{2n\pi/(\beta-\alpha)}\right]\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right] \\ &\quad + \sum_{n=1}^\infty B_n r^{n\pi/(\beta-\alpha)}\left[1-\left(\frac{a}{r}\right)^{2n\pi/(\beta-\alpha)}\right]\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right] \end{aligned}$$

Therefore,

$$\boxed{u(r,\theta) = \sum_{n=1}^\infty \left\{A_n\left[1-\left(\frac{b}{r}\right)^{2n\pi/(\beta-\alpha)}\right] + B_n\left[1-\left(\frac{a}{r}\right)^{2n\pi/(\beta-\alpha)}\right]\right\} r^{n\pi/(\beta-\alpha)}\sin\left[\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right].}$$