Exercise 3

Determine the coefficients in the annulus problem of the text.

Solution

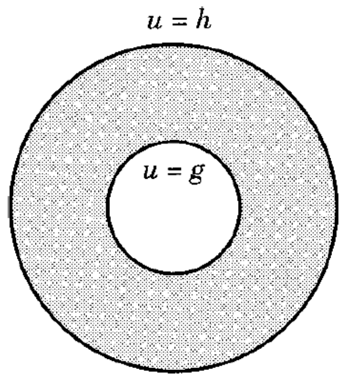

Figure 2

The annulus problem of the text is shown in Figure 2 and stated as follows.

$$\begin{aligned} u_{xx} + u_{yy} &= 0 && \text{in } 0 < a^2 < x^2 + y^2 < b^2 \\ u &= g(\theta) && \text{for } x^2 + y^2 = a^2 \\ u &= h(\theta) && \text{for } x^2 + y^2 = b^2 \end{aligned}$$

Its solution is

$$u(r,\theta) = \frac{1}{2}(C_0 + D_0 \ln r) + \sum_{n=1}^{\infty}[(C_n r^n + D_n r^{-n})\cos n\theta + (A_n r^n + B_n r^{-n})\sin n\theta].$$

Apply the inhomogeneous boundary conditions at $r = a$ and $r = b$ to determine the coefficients. Since there are six coefficients, we will need six equations to solve for them.

$$u(a,\theta) = \frac{1}{2}(C_0 + D_0 \ln a) + \sum_{n=1}^{\infty}[(C_n a^n + D_n a^{-n})\cos n\theta + (A_n a^n + B_n a^{-n})\sin n\theta] = g(\theta) \quad (1)$$

$$u(b,\theta) = \frac{1}{2}(C_0 + D_0 \ln b) + \sum_{n=1}^{\infty}[(C_n b^n + D_n b^{-n})\cos n\theta + (A_n b^n + B_n b^{-n})\sin n\theta] = h(\theta) \quad (2)$$

Integrate both sides of equations (1) and (2) with respect to θ from 0 to 2π to obtain the first two equations.

$$\int_0^{2\pi}\left\{\frac{1}{2}(C_0 + D_0 \ln a) + \sum_{n=1}^{\infty}[(C_n a^n + D_n a^{-n})\cos n\theta + (A_n a^n + B_n a^{-n})\sin n\theta]\right\}d\theta = \int_0^{2\pi} g(\theta)\,d\theta$$

$$\int_0^{2\pi}\left\{\frac{1}{2}(C_0 + D_0 \ln b) + \sum_{n=1}^{\infty}[(C_n b^n + D_n b^{-n})\cos n\theta + (A_n b^n + B_n b^{-n})\sin n\theta]\right\}d\theta = \int_0^{2\pi} h(\theta)\,d\theta$$

Split the integrals on the left into three and bring the constants in front of them.

$$\frac{1}{2}(C_0 + D_0 \ln a)\int_0^{2\pi} d\theta + \sum_{n=1}^{\infty}\left[(C_n a^n + D_n a^{-n})\underbrace{\int_0^{2\pi}\cos n\theta\, d\theta}_{=0} + (A_n a^n + B_n a^{-n})\underbrace{\int_0^{2\pi}\sin n\theta\, d\theta}_{=0}\right] = \int_0^{2\pi} g(\theta)\, d\theta$$

$$\frac{1}{2}(C_0 + D_0 \ln b)\int_0^{2\pi} d\theta + \sum_{n=1}^{\infty}\left[(C_n b^n + D_n b^{-n})\underbrace{\int_0^{2\pi}\cos n\theta\, d\theta}_{=0} + (A_n b^n + B_n b^{-n})\underbrace{\int_0^{2\pi}\sin n\theta\, d\theta}_{=0}\right] = \int_0^{2\pi} h(\theta)\, d\theta$$

Evaluate the remaining integrals on the left side.

$$\frac{1}{2}(C_0 + D_0 \ln a)\cdot 2\pi = \int_0^{2\pi} g(\theta)\, d\theta \quad \to \quad C_0 + D_0 \ln a = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\, d\theta$$

$$\frac{1}{2}(C_0 + D_0 \ln b)\cdot 2\pi = \int_0^{2\pi} h(\theta)\, d\theta \quad \to \quad C_0 + D_0 \ln b = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\, d\theta \tag{3}$$

This is a system of two equations for C_0 and D_0. Subtract both sides of the second equation from those of the first one to eliminate C_0.

$$D_0 \ln a - D_0 \ln b = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\, d\theta - \frac{1}{\pi}\int_0^{2\pi} h(\theta)\, d\theta$$

Solve for D_0.

$$\boxed{D_0 = \frac{1}{\pi \ln \frac{b}{a}}\int_0^{2\pi} [h(\theta) - g(\theta)]\, d\theta}$$

Substitute this result into equation (3) to solve for C_0.

$$C_0 + \frac{\ln b}{\pi \ln \frac{b}{a}}\int_0^{2\pi} [h(\theta) - g(\theta)]\, d\theta = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\, d\theta$$

Solve for C_0.

$$\begin{aligned} C_0 &= \left(1 - \frac{\ln b}{\ln \frac{b}{a}}\right)\frac{1}{\pi}\int_0^{2\pi} h(\theta)\, d\theta + \frac{\ln b}{\pi \ln \frac{b}{a}}\int_0^{2\pi} g(\theta)\, d\theta \\ &= \left(-\frac{\ln a}{\ln \frac{b}{a}}\right)\frac{1}{\pi}\int_0^{2\pi} h(\theta)\, d\theta + \frac{\ln b}{\pi \ln \frac{b}{a}}\int_0^{2\pi} g(\theta)\, d\theta \\ &= \frac{1}{\pi \ln \frac{b}{a}}\int_0^{2\pi} (-\ln a) h(\theta)\, d\theta + \frac{1}{\pi \ln \frac{b}{a}}\int_0^{2\pi} (\ln b) g(\theta)\, d\theta \end{aligned}$$

Therefore,

$$\boxed{C_0 = \frac{1}{\pi \ln \frac{b}{a}}\int_0^{2\pi} [g(\theta)\ln b - h(\theta)\ln a]\, d\theta.}$$

To obtain another system of two equations for C_n and D_n, multiply both sides of equations (1) and (2) by $\cos m\theta$, where m is an integer,

$$\frac{1}{2}(C_0 + D_0 \ln a)\cos m\theta + \sum_{n=1}^{\infty}[(C_n a^n + D_n a^{-n})\cos n\theta \cos m\theta + (A_n a^n + B_n a^{-n})\sin n\theta \cos m\theta] = g(\theta)\cos m\theta$$

$$\frac{1}{2}(C_0 + D_0 \ln b)\cos m\theta + \sum_{n=1}^{\infty}[(C_n b^n + D_n b^{-n})\cos n\theta \cos m\theta + (A_n b^n + B_n b^{-n})\sin n\theta \cos m\theta] = h(\theta)\cos m\theta$$

and then integrate both sides of each with respect to θ from 0 to 2π.

$$\int_0^{2\pi} \left\{ \frac{1}{2}(C_0 + D_0 \ln a)\cos m\theta + \sum_{n=1}^{\infty} [(C_n a^n + D_n a^{-n})\cos n\theta \cos m\theta + (A_n a^n + B_n a^{-n})\sin n\theta \cos m\theta] \right\} d\theta = \int_0^{2\pi} g(\theta)\cos m\theta \, d\theta$$

$$\int_0^{2\pi} \left\{ \frac{1}{2}(C_0 + D_0 \ln b)\cos m\theta + \sum_{n=1}^{\infty} [(C_n b^n + D_n b^{-n})\cos n\theta \cos m\theta + (A_n b^n + B_n b^{-n})\sin n\theta \cos m\theta] \right\} d\theta = \int_0^{2\pi} h(\theta)\cos m\theta \, d\theta$$

Split the integrals on the left into three each and bring the constants in front of them.

$$\frac{1}{2}(C_0 + D_0 \ln a) \underbrace{\int_0^{2\pi} \cos m\theta \, d\theta}_{=0} + \sum_{n=1}^{\infty} \left[(C_n a^n + D_n a^{-n}) \int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta + (A_n a^n + B_n a^{-n}) \underbrace{\int_0^{2\pi} \sin n\theta \cos m\theta \, d\theta}_{=0} \right] = \int_0^{2\pi} g(\theta)\cos m\theta \, d\theta$$

$$\frac{1}{2}(C_0 + D_0 \ln b) \underbrace{\int_0^{2\pi} \cos m\theta \, d\theta}_{=0} + \sum_{n=1}^{\infty} \left[(C_n b^n + D_n b^{-n}) \int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta + (A_n b^n + B_n b^{-n}) \underbrace{\int_0^{2\pi} \sin n\theta \cos m\theta \, d\theta}_{=0} \right] = \int_0^{2\pi} h(\theta)\cos m\theta \, d\theta$$

Sine and cosine are orthogonal for all integer values of n and m, so the third integrals on the left are equal to zero.

$$\sum_{n=1}^{\infty} (C_n a^n + D_n a^{-n}) \int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta = \int_0^{2\pi} g(\theta)\cos m\theta \, d\theta$$
$$\sum_{n=1}^{\infty} (C_n b^n + D_n b^{-n}) \int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta = \int_0^{2\pi} h(\theta)\cos m\theta \, d\theta$$

Because the cosine functions are orthogonal, the remaining integrals on the left are zero for $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$(C_n a^n + D_n a^{-n}) \int_0^{2\pi} \cos^2 n\theta \, d\theta = \int_0^{2\pi} g(\theta)\cos n\theta \, d\theta$$
$$(C_n b^n + D_n b^{-n}) \int_0^{2\pi} \cos^2 n\theta \, d\theta = \int_0^{2\pi} h(\theta)\cos n\theta \, d\theta$$

Evaluate the integrals on the left.

$$(C_n a^n + D_n a^{-n}) \cdot \pi = \int_0^{2\pi} g(\theta)\cos n\theta\, d\theta \quad \rightarrow \quad C_n a^n + D_n a^{-n} = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta\, d\theta \tag{4}$$

$$(C_n b^n + D_n b^{-n}) \cdot \pi = \int_0^{2\pi} h(\theta)\cos n\theta\, d\theta \quad \rightarrow \quad C_n b^n + D_n b^{-n} = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos n\theta\, d\theta \tag{5}$$

To solve for C_n, multiply both sides of equation (4) by a^n, multiply both sides of equation (5) by b^n,

$$C_n a^{2n} + D_n = \frac{a^n}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta\, d\theta$$

$$C_n b^{2n} + D_n = \frac{b^n}{\pi}\int_0^{2\pi} h(\theta)\cos n\theta\, d\theta$$

and subtract both sides of the second equation from those of the first one.

$$C_n a^{2n} - C_n b^{2n} = \frac{a^n}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta\, d\theta - \frac{b^n}{\pi}\int_0^{2\pi} h(\theta)\cos n\theta\, d\theta$$

Factor C_n and bring a^n and b^n inside the integrands in order to combine the integrals.

$$\begin{aligned} C_n(a^{2n} - b^{2n}) &= \frac{1}{\pi}\int_0^{2\pi} a^n g(\theta)\cos n\theta\, d\theta - \frac{1}{\pi}\int_0^{2\pi} b^n h(\theta)\cos n\theta\, d\theta \\ &= \frac{1}{\pi}\int_0^{2\pi} [a^n g(\theta)\cos n\theta - b^n h(\theta)\cos n\theta]\, d\theta \end{aligned}$$

Therefore,

$$\boxed{C_n = \frac{1}{\pi(a^{2n} - b^{2n})}\int_0^{2\pi} [a^n g(\theta) - b^n h(\theta)]\cos n\theta\, d\theta.}$$

To solve for D_n, divide both sides of equation (4) by a^n, divide both sides of equation (5) by b^n,

$$C_n + D_n a^{-2n} = \frac{a^{-n}}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta\, d\theta$$

$$C_n + D_n b^{-2n} = \frac{b^{-n}}{\pi}\int_0^{2\pi} h(\theta)\cos n\theta\, d\theta$$

and subtract both sides of the second equation from those of the first one.

$$D_n a^{-2n} - D_n b^{-2n} = \frac{a^{-n}}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta\, d\theta - \frac{b^{-n}}{\pi}\int_0^{2\pi} h(\theta)\cos n\theta\, d\theta$$

Factor D_n and bring a^{-n} and b^{-n} inside the integrands in order to combine the integrals.

$$\begin{aligned} D_n(a^{-2n} - b^{-2n}) &= \frac{1}{\pi}\int_0^{2\pi} a^{-n} g(\theta)\cos n\theta\, d\theta - \frac{1}{\pi}\int_0^{2\pi} b^{-n} h(\theta)\cos n\theta\, d\theta \\ &= \frac{1}{\pi}\int_0^{2\pi} [a^{-n} g(\theta)\cos n\theta - b^{-n} h(\theta)\cos n\theta]\, d\theta \end{aligned}$$

Therefore,

$$\boxed{D_n = \frac{1}{\pi(a^{-2n} - b^{-2n})} \int_0^{2\pi} [a^{-n} g(\theta) - b^{-n} h(\theta)] \cos n\theta \, d\theta.}$$

To obtain another system of two equations for A_n and B_n, multiply both sides of equations (1) and (2) by $\sin m\theta$, where m is an integer,

$$\frac{1}{2}(C_0 + D_0 \ln a) \sin m\theta + \sum_{n=1}^{\infty} [(C_n a^n + D_n a^{-n}) \cos n\theta \sin m\theta + (A_n a^n + B_n a^{-n}) \sin n\theta \sin m\theta] = g(\theta) \sin m\theta$$

$$\frac{1}{2}(C_0 + D_0 \ln b) \sin m\theta + \sum_{n=1}^{\infty} [(C_n b^n + D_n b^{-n}) \cos n\theta \sin m\theta + (A_n b^n + B_n b^{-n}) \sin n\theta \sin m\theta] = h(\theta) \sin m\theta$$

and then integrate both sides of each with respect to θ from 0 to 2π.

$$\int_0^{2\pi} \left\{ \frac{1}{2}(C_0 + D_0 \ln a) \sin m\theta + \sum_{n=1}^{\infty} [(C_n a^n + D_n a^{-n}) \cos n\theta \sin m\theta + (A_n a^n + B_n a^{-n}) \sin n\theta \sin m\theta] \right\} d\theta = \int_0^{2\pi} g(\theta) \sin m\theta \, d\theta$$

$$\int_0^{2\pi} \left\{ \frac{1}{2}(C_0 + D_0 \ln b) \sin m\theta + \sum_{n=1}^{\infty} [(C_n b^n + D_n b^{-n}) \cos n\theta \sin m\theta + (A_n b^n + B_n b^{-n}) \sin n\theta \sin m\theta] \right\} d\theta = \int_0^{2\pi} h(\theta) \sin m\theta \, d\theta$$

Split the integrals on the left into three each and bring the constants in front of them.

$$\frac{1}{2}(C_0 + D_0 \ln a) \underbrace{\int_0^{2\pi} \sin m\theta \, d\theta}_{=0} + \sum_{n=1}^{\infty} \left[(C_n a^n + D_n a^{-n}) \underbrace{\int_0^{2\pi} \cos n\theta \sin m\theta \, d\theta}_{=0} + (A_n a^n + B_n a^{-n}) \int_0^{2\pi} \sin n\theta \sin m\theta \, d\theta \right] = \int_0^{2\pi} g(\theta) \sin m\theta \, d\theta$$

$$\frac{1}{2}(C_0 + D_0 \ln b) \underbrace{\int_0^{2\pi} \sin m\theta \, d\theta}_{=0} + \sum_{n=1}^{\infty} \left[(C_n b^n + D_n b^{-n}) \underbrace{\int_0^{2\pi} \cos n\theta \sin m\theta \, d\theta}_{=0} + (A_n b^n + B_n b^{-n}) \int_0^{2\pi} \sin n\theta \sin m\theta \, d\theta \right] = \int_0^{2\pi} h(\theta) \sin m\theta \, d\theta$$

Sine and cosine are orthogonal for all integer values of n and m, so the second integrals on the left are equal to zero.

$$\sum_{n=1}^{\infty} (A_n a^n + B_n a^{-n}) \int_0^{2\pi} \sin n\theta \sin m\theta \, d\theta = \int_0^{2\pi} g(\theta) \sin m\theta \, d\theta$$

$$\sum_{n=1}^{\infty} (A_n b^n + B_n b^{-n}) \int_0^{2\pi} \sin n\theta \sin m\theta \, d\theta = \int_0^{2\pi} h(\theta) \sin m\theta \, d\theta$$

Because the sine functions are orthogonal, the remaining integrals on the left are zero for $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$\begin{aligned}(A_n a^n + B_n a^{-n}) \int_0^{2\pi} \sin^2 n\theta \, d\theta &= \int_0^{2\pi} g(\theta) \sin n\theta \, d\theta \\ (A_n b^n + B_n b^{-n}) \int_0^{2\pi} \sin^2 n\theta \, d\theta &= \int_0^{2\pi} h(\theta) \sin n\theta \, d\theta\end{aligned}$$

Evaluate the integrals on the left.

$$(A_n a^n + B_n a^{-n}) \cdot \pi = \int_0^{2\pi} g(\theta) \sin n\theta \, d\theta \quad \rightarrow \quad A_n a^n + B_n a^{-n} = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \sin n\theta \, d\theta \tag{6}$$

$$(A_n b^n + B_n b^{-n}) \cdot \pi = \int_0^{2\pi} h(\theta) \sin n\theta \, d\theta \quad \rightarrow \quad A_n b^n + B_n b^{-n} = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \sin n\theta \, d\theta \tag{7}$$

To solve for A_n, multiply both sides of equation (6) by a^n, multiply both sides of equation (7) by b^n,

$$\begin{aligned}A_n a^{2n} + B_n &= \frac{a^n}{\pi} \int_0^{2\pi} g(\theta) \sin n\theta \, d\theta \\ A_n b^{2n} + B_n &= \frac{b^n}{\pi} \int_0^{2\pi} h(\theta) \sin n\theta \, d\theta\end{aligned}$$

and subtract both sides of the second equation from those of the first one.

$$A_n a^{2n} - A_n b^{2n} = \frac{a^n}{\pi} \int_0^{2\pi} g(\theta) \sin n\theta \, d\theta - \frac{b^n}{\pi} \int_0^{2\pi} h(\theta) \sin n\theta \, d\theta$$

Factor A_n and bring a^n and b^n inside the integrands in order to combine the integrals.

$$\begin{aligned}A_n(a^{2n} - b^{2n}) &= \frac{1}{\pi} \int_0^{2\pi} a^n g(\theta) \sin n\theta \, d\theta - \frac{1}{\pi} \int_0^{2\pi} b^n h(\theta) \sin n\theta \, d\theta \\ &= \frac{1}{\pi} \int_0^{2\pi} [a^n g(\theta) \sin n\theta - b^n h(\theta) \sin n\theta] \, d\theta\end{aligned}$$

Therefore,

$$\boxed{A_n = \frac{1}{\pi(a^{2n} - b^{2n})} \int_0^{2\pi} [a^n g(\theta) - b^n h(\theta)] \sin n\theta \, d\theta.}$$

To solve for B_n, divide both sides of equation (6) by a^n, divide both sides of equation (7) by b^n,

$$\begin{aligned}A_n + B_n a^{-2n} &= \frac{a^{-n}}{\pi} \int_0^{2\pi} g(\theta) \sin n\theta \, d\theta \\ A_n + B_n b^{-2n} &= \frac{b^{-n}}{\pi} \int_0^{2\pi} h(\theta) \sin n\theta \, d\theta\end{aligned}$$

and subtract both sides of the second equation from those of the first one.

$$B_n a^{-2n} - B_n b^{-2n} = \frac{a^{-n}}{\pi} \int_0^{2\pi} g(\theta) \sin \theta \, d\theta - \frac{b^{-n}}{\pi} \int_0^{2\pi} h(\theta) \sin n\theta \, d\theta$$

Factor B_n and bring a^{-n} and b^{-n} inside the integrands in order to combine the integrals.

$$\begin{aligned} B_n(a^{-2n} - b^{-2n}) &= \frac{1}{\pi}\int_0^{2\pi} a^{-n} g(\theta)\sin n\theta\,d\theta - \frac{1}{\pi}\int_0^{2\pi} b^{-n} h(\theta)\sin n\theta\,d\theta \\ &= \frac{1}{\pi}\int_0^{2\pi} [a^{-n} g(\theta)\sin n\theta - b^{-n} h(\theta)\sin n\theta]\,d\theta \end{aligned}$$

Therefore,

$$\boxed{B_n = \frac{1}{\pi(a^{-2n} - b^{-2n})}\int_0^{2\pi} [a^{-n} g(\theta) - b^{-n} h(\theta)]\sin n\theta\,d\theta.}$$