Exercise 4

Derive Poisson's formula (9) for the exterior of a circle.

Solution

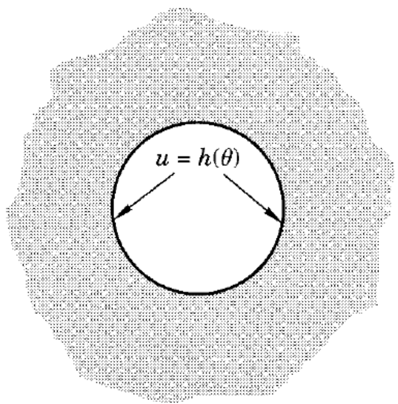

Figure 3

The Dirichlet problem for the exterior of a circle (see Figure 3) is

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \qquad \text{for } x^2 + y^2 > a^2 \\ u &= h(\theta) \qquad \text{for } x^2 + y^2 = a^2 \end{aligned}$$

$$u \text{ bounded as } x^2 + y^2 \to \infty.$$

Since the boundary condition is given on a circle, we choose to reformulate the problem in polar coordinates. We also expect the solution to be the same in value and slope (in the θ-direction) at $\theta = 0$ as it is at $\theta = 2\pi$.

$$\begin{gathered} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad a < r < \infty,\ 0 < \theta < 2\pi \\ u(a, \theta) = h(\theta), \quad u(r, 0) = u(r, 2\pi) \\ u(\infty, \theta) = \text{bounded}, \quad u_\theta(r, 0) = u_\theta(r, 2\pi). \end{gathered}$$

The Laplace equation and all but one of its associated boundary conditions are linear and homogeneous, so the method of separation of variables can be applied to solve the problem. Assume a product solution of the form $u = R(r)\Theta(\theta)$ and plug it into the PDE

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \to \quad R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

and the homogeneous boundary conditions.

$$\begin{aligned} u(r, 0) = u(r, 2\pi) \quad &\to \quad R(r)\Theta(0) = R(r)\Theta(2\pi) \quad &\to \quad \Theta(0) = \Theta(2\pi) \\ u_\theta(r, 0) = u_\theta(r, 2\pi) \quad &\to \quad R(r)\Theta'(0) = R(r)\Theta'(2\pi) \quad &\to \quad \Theta'(0) = \Theta'(2\pi) \\ u(\infty, \theta) = \text{bounded} \quad &\to \quad R(\infty)\Theta(\theta) = \text{bounded} \quad &\to \quad R(\infty) = \text{bounded} \end{aligned}$$

Now separate variables in the PDE: bring all functions of r to the left side and all functions of θ to the right side. Note that the final answer will be the same regardless of which side the minus sign is on.

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \quad \rightarrow \quad r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0 \quad \rightarrow \quad r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta}$$

The only way that a function of r can be equal to a function of θ is if both are equal to a constant λ.

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

Values of λ for which the boundary conditions are satisfied are called the eigenvalues, and the nontrivial functions associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

If λ is positive, then the ODE for Θ becomes

$$-\frac{\Theta''}{\Theta} = \mu^2.$$

Multiply both sides by $-\Theta$.

$$\Theta'' = -\mu^2\Theta$$

The general solution can be written in terms of sine and cosine.

$$\Theta(\theta) = C_1 \cos\mu\theta + C_2 \sin\mu\theta$$

Take a derivative of it with respect to θ.

$$\Theta'(\theta) = \mu(-C_1 \sin\mu\theta + C_2 \cos\mu\theta)$$

Apply the boundary conditions here to determine C_1 and C_2.

$$\Theta(0) = C_1 = C_1 \cos 2\pi\mu + C_2 \sin 2\pi\mu = \Theta(2\pi)$$
$$\Theta'(0) = \mu(C_2) = \mu(-C_1 \sin 2\pi\mu + C_2 \cos 2\pi\mu) = \Theta'(2\pi)$$

Solve the first equation for C_1 and divide both sides of the second equation by μ.

$$C_1 = C_2\frac{\sin 2\pi\mu}{1 - \cos 2\pi\mu}$$
$$C_2 = -C_1 \sin 2\pi\mu + C_2 \cos 2\pi\mu$$

Substitute the formula for C_1 into the second equation.

$$C_2 = -C_2\frac{\sin 2\pi\mu}{1 - \cos 2\pi\mu}\sin 2\pi\mu + C_2 \cos 2\pi\mu$$

Multiply both sides by $(1 - \cos 2\pi\mu)/C_2$.

$$1 - \cos 2\pi\mu = -\sin^2 2\pi\mu + \cos 2\pi\mu - \cos^2 2\pi\mu$$

Simplify the equation.

$$1 = \cos 2\pi\mu$$

This equation only holds if $2\pi\mu$ is an integer multiple of 2π.

$$\begin{aligned} 2\pi\mu &= 2\pi n \\ \mu_n &= n, \quad n = 1, 2, \ldots \end{aligned}$$

The eigenfunctions corresponding with these eigenvalues for λ are

$$\Theta(\theta) = C_1 \cos \mu\theta + C_2 \sin \mu\theta \quad \to \quad \Theta_n(\theta) = C_1 \cos n\theta + C_2 \sin n\theta, \quad n = 1, 2, \ldots.$$

The ODE for $R(r)$ will now be solved.

$$r^2 \frac{R''}{R} + r\frac{R'}{R} = \mu^2$$

Bring μ^2 to the left side and multiply both sides by R.

$$r^2 R'' + rR' - \mu^2 R = 0$$

This is an equidimensional ODE, so the solutions are of the form, $R(r) = r^m$. Find the first and second derivatives of it

$$R = r^m \quad \to \quad R' = mr^{m-1} \quad \to \quad R'' = m(m-1)r^{m-2}$$

and then substitute these formulas into the ODE to determine the values of m.

$$m(m-1)r^m + mr^m - \mu^2 r^m = 0$$

$$\begin{aligned} m(m-1) + m - \mu^2 &= 0 \\ m^2 - \mu^2 &= 0 \\ m &= \{\pm\mu\} \end{aligned}$$

So then

$$R(r) = C_3 r^{\mu} + C_4 r^{-\mu}.$$

For the boundary condition, $R(\infty) =$ bounded, to be satisfied, we require $C_3 = 0$.

$$R(r) = C_4 r^{-\mu} \quad \to \quad R_n(r) = r^{-n}$$

Determination of the Zero Eigenvalue: $\lambda = 0$

If λ is zero, then the ODE for Θ becomes

$$-\frac{\Theta''}{\Theta} = 0.$$

Multiply both sides by $-\Theta$.

$$\Theta''(\theta) = 0$$

Integrate both sides with respect to θ.

$$\Theta'(\theta) = C_5$$

Integrate both sides with respect to θ once more.

$$\Theta(\theta) = C_5\theta + C_6$$

Apply the boundary conditions here to determine C_5 and C_6.

$$\Theta(0) = C_6 = 2\pi C_5 + C_6 = \Theta(2\pi)$$
$$\Theta'(0) = C_5 = C_5 = \Theta'(2\pi)$$

From the first equation we find that $C_5 = 0$.

$$\Theta(\theta) = C_6$$

Now the equation for $R(r)$ will be solved.

$$r^2 \frac{R''}{R} + r\frac{R'}{R} = 0$$

Multiply both sides by R/r.

$$rR'' + R' = 0$$

The left side can be written as $d/dr(rR')$ by the product rule.

$$\frac{d}{dr}(rR') = 0$$

Integrate both sides with respect to r.

$$rR' = C_7$$

Divide both sides by r.

$$R' = \frac{C_7}{r}$$

Integrate both sides with respect to r.

$$R(r) = C_7 \ln r + C_8$$

For the boundary condition, $R(\infty) =$ bounded, to be satisfied, we require $C_7 = 0$. To conclude, zero is an eigenvalue for λ, and the eigenfunctions associated with it are constants.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

If λ is negative, then the ODE for Θ becomes

$$-\frac{\Theta''}{\Theta} = -\gamma^2.$$

Multiply both sides by $-\Theta$.

$$\Theta'' = \gamma^2\Theta$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_9 \cosh\gamma\theta + C_{10} \sinh\gamma\theta$$

Take a derivative of this solution.

$$\Theta'(\theta) = \gamma(C_9 \sinh\gamma\theta + C_{10} \cosh\gamma\theta)$$

Apply the boundary conditions here to find C_9 and C_{10}.

$$\Theta(0) = C_9 = C_9 \cosh 2\pi\gamma + C_{10} \sinh 2\pi\gamma = \Theta(2\pi)$$
$$\Theta'(0) = \gamma(C_{10}) = \gamma(C_9 \sinh 2\pi\gamma + C_{10} \cosh 2\pi\gamma) = \Theta'(2\pi)$$

Solve the first equation for C_9 and divide both sides of the second equation by γ.

$$C_9 = C_{10} \frac{\sinh 2\pi\gamma}{1 - \cosh 2\pi\gamma}$$
$$C_{10} = C_9 \sinh 2\pi\gamma + C_{10} \cosh 2\pi\gamma$$

Substitute the formula for C_9 into the second equation.

$$C_{10} = C_{10} \frac{\sinh 2\pi\gamma}{1 - \cosh 2\pi\gamma} \sinh 2\pi\gamma + C_{10} \cosh 2\pi\gamma$$

Multiply both sides by $(1 - \cosh 2\pi\gamma)/C_{10}$.

$$1 - \cosh 2\pi\gamma = \sinh^2 2\pi\gamma + \cosh 2\pi\gamma - \cosh^2 2\pi\gamma$$

Simplify the equation.

$$1 = \cosh 2\pi\gamma$$

Because hyperbolic cosine is not oscillatory, there are no values of γ that satisfy this equation. Hence, there are no negative eigenvalues for λ.

According to the principle of superposition, the general solution for u is a linear combination of the eigenfunctions $R(r)\Theta(\theta)$ over all the eigenvalues.

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n \cos n\theta + B_n \sin n\theta) \tag{1}$$

To determine the coefficients, A_0, A_n, and B_n, we apply the inhomogeneous boundary condition at $r = a$.

$$u(a, \theta) = A_0 + \sum_{n=1}^{\infty} a^{-n}(A_n \cos n\theta + B_n \sin n\theta) = h(\theta) \tag{2}$$

Integrate both sides of equation (2) with respect to θ from 0 to 2π to obtain A_0.

$$\int_0^{2\pi} \left[A_0 + \sum_{n=1}^{\infty} a^{-n}(A_n \cos n\theta + B_n \sin n\theta)\right] d\theta = \int_0^{2\pi} h(\theta)\, d\theta$$

Split up the integral on the left into three and bring the constants in front of them.

$$A_0 \int_0^{2\pi} d\theta + \sum_{n=1}^{\infty} a^{-n}\left(A_n \underbrace{\int_0^{2\pi} \cos n\theta\, d\theta}_{=0} + B_n \underbrace{\int_0^{2\pi} \sin n\theta\, d\theta}_{=0}\right) = \int_0^{2\pi} h(\theta)\, d\theta$$

Evaluate the remaining integral on the left side.

$$A_0 \cdot 2\pi = \int_0^{2\pi} h(\theta)\, d\theta$$

Solve for A_0.

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta \tag{3}$$

To obtain A_n, multiply both sides of equation (2) by $\cos m\theta$, where m is an integer,

$$A_0 \cos m\theta + \sum_{n=1}^{\infty} a^{-n}(A_n \cos n\theta \cos m\theta + B_n \sin n\theta \cos m\theta) = h(\theta)\cos m\theta$$

and then integrate both sides with respect to θ from 0 to 2π.

$$\int_0^{2\pi}\left[A_0 \cos m\theta + \sum_{n=1}^{\infty} a^{-n}(A_n \cos n\theta \cos m\theta + B_n \sin n\theta \cos m\theta)\right] d\theta = \int_0^{2\pi} h(\theta)\cos m\theta\,d\theta$$

Split up the integral on the left into three and bring the constants in front of them.

$$A_0 \underbrace{\int_0^{2\pi} \cos m\theta\,d\theta}_{=0} + \sum_{n=1}^{\infty} a^{-n}\left(A_n \int_0^{2\pi} \cos n\theta \cos m\theta\,d\theta + B_n \underbrace{\int_0^{2\pi} \sin n\theta \cos m\theta\,d\theta}_{=0}\right) = \int_0^{2\pi} h(\theta)\cos m\theta\,d\theta$$

Sine and cosine are orthogonal for all integer values of n and m, so the third integral on the left is zero.

$$\sum_{n=1}^{\infty} a^{-n} A_n \int_0^{2\pi} \cos n\theta \cos m\theta\,d\theta = \int_0^{2\pi} h(\theta)\cos m\theta\,d\theta$$

Because the cosine functions are orthogonal, the remaining integral on the left is zero for $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$a^{-n} A_n \int_0^{2\pi} \cos^2 n\theta\,d\theta = \int_0^{2\pi} h(\theta)\cos n\theta\,d\theta$$

Evaluate the last integral on the left side.

$$a^{-n} A_n \cdot \pi = \int_0^{2\pi} h(\theta)\cos n\theta\,d\theta$$

Solve for A_n.

$$A_n = \frac{a^n}{\pi}\int_0^{2\pi} h(\theta)\cos n\theta\,d\theta \tag{4}$$

To obtain B_n, multiply both sides of equation (2) by $\sin m\theta$, where m is an integer,

$$A_0 \sin m\theta + \sum_{n=1}^{\infty} a^{-n}(A_n \cos n\theta \sin m\theta + B_n \sin n\theta \sin m\theta) = h(\theta)\sin m\theta$$

and then integrate both sides with respect to θ from 0 to 2π.

$$\int_0^{2\pi}\left[A_0 \sin m\theta + \sum_{n=1}^{\infty} a^{-n}(A_n \cos n\theta \sin m\theta + B_n \sin n\theta \sin m\theta)\right] d\theta = \int_0^{2\pi} h(\theta)\sin m\theta\,d\theta$$

Split up the integral on the left into three and bring the constants in front of them.

$$A_0 \underbrace{\int_0^{2\pi} \sin m\theta\,d\theta}_{=0} + \sum_{n=1}^{\infty} a^{-n}\left(A_n \underbrace{\int_0^{2\pi} \cos n\theta \sin m\theta\,d\theta}_{=0} + B_n \int_0^{2\pi} \sin n\theta \sin m\theta\,d\theta\right) = \int_0^{2\pi} h(\theta)\sin m\theta\,d\theta$$

Sine and cosine are orthogonal for all integer values of n and m, so the second integral on the left is zero.

$$\sum_{n=1}^{\infty} a^{-n} B_n \int_0^{2\pi} \sin n\theta \sin m\theta \, d\theta = \int_0^{2\pi} h(\theta) \sin m\theta \, d\theta$$

Because the sine functions are orthogonal, the remaining integral on the left is zero for $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$a^{-n} B_n \int_0^{2\pi} \sin^2 n\theta \, d\theta = \int_0^{2\pi} h(\theta) \sin n\theta \, d\theta$$

Evaluate the last integral on the left side.

$$a^{-n} B_n \cdot \pi = \int_0^{2\pi} h(\theta) \sin n\theta \, d\theta$$

Solve for B_n.

$$B_n = \frac{a^n}{\pi} \int_0^{2\pi} h(\theta) \sin n\theta \, d\theta \tag{5}$$

Now substitute equations (3), (4), and (5) into equation (1). The dummy variable ϕ is used for the integration variable.

$$\begin{aligned}
u(r,\theta) &= A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n \cos n\theta + B_n \sin n\theta) \\
&= \frac{1}{2\pi} \int_0^{2\pi} h(\phi)\, d\phi + \sum_{n=1}^{\infty} r^{-n} \left\{ \left[\frac{a^n}{\pi} \int_0^{2\pi} h(\phi) \cos n\phi \, d\phi \right] \cos n\theta + \left[\frac{a^n}{\pi} \int_0^{2\pi} h(\phi) \sin n\phi \, d\phi \right] \sin n\theta \right\} \\
&= \frac{1}{2\pi} \int_0^{2\pi} h(\phi)\, d\phi + \sum_{n=1}^{\infty} \frac{a^n}{\pi r^n} \left[\int_0^{2\pi} h(\phi) \cos n\theta \cos n\phi \, d\phi + \int_0^{2\pi} h(\phi) \sin n\theta \sin n\phi \, d\phi \right] \\
&= \frac{1}{2\pi} \int_0^{2\pi} h(\phi)\, d\phi + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \int_0^{2\pi} (\cos n\theta \cos n\phi + \sin n\theta \sin n\phi) h(\phi)\, d\phi \\
&= \int_0^{2\pi} \left[\frac{h(\phi)}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \cos(n\theta - n\phi) h(\phi) \right] d\phi \\
&= \int_0^{2\pi} \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \cos[n(\theta - \phi)] \right] \frac{h(\phi)}{2\pi}\, d\phi \\
&= \int_0^{2\pi} \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \frac{e^{in(\theta-\phi)} + e^{-in(\theta-\phi)}}{2} \right] \frac{h(\phi)}{2\pi}\, d\phi \\
&= \int_0^{2\pi} \left[1 + \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n e^{-in(\theta-\phi)} \right] \frac{h(\phi)}{2\pi}\, d\phi \\
&= \int_0^{2\pi} \left[-1 + \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^n e^{in(\theta-\phi)} + \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^n e^{-in(\theta-\phi)} \right] \frac{h(\phi)}{2\pi}\, d\phi \\
&= \int_0^{2\pi} \left\{ -1 + \sum_{n=0}^{\infty} \left[\left(\frac{a}{r}\right) e^{i(\theta-\phi)} \right]^n + \sum_{n=0}^{\infty} \left[\left(\frac{a}{r}\right) e^{-i(\theta-\phi)} \right]^n \right\} \frac{h(\phi)}{2\pi}\, d\phi
\end{aligned}$$

Recall that an infinite geometric series can be summed as

$$\sum_{n=0}^{\infty} b^n = \frac{1}{1-b},$$

assuming $|b| < 1$. Applying this to the formula for u is justified because $r > a$. We get

$$\begin{aligned}
u(r,\theta) &= \int_0^{2\pi} \left\{ -1 + \frac{1}{1 - \left(\frac{a}{r}\right) e^{i(\theta-\phi)}} + \frac{1}{1 - \left(\frac{a}{r}\right) e^{-i(\theta-\phi)}} \right\} \frac{h(\phi)}{2\pi}\, d\phi \\
&= \int_0^{2\pi} \frac{-\left[1 - \left(\frac{a}{r}\right) e^{i(\theta-\phi)}\right]\left[1 - \left(\frac{a}{r}\right) e^{-i(\theta-\phi)}\right] + 1 - \left(\frac{a}{r}\right) e^{-i(\theta-\phi)} + 1 - \left(\frac{a}{r}\right) e^{i(\theta-\phi)}}{\left[1 - \left(\frac{a}{r}\right) e^{i(\theta-\phi)}\right]\left[1 - \left(\frac{a}{r}\right) e^{-i(\theta-\phi)}\right]} \frac{h(\phi)}{2\pi}\, d\phi \\
&= \int_0^{2\pi} \frac{1 - \frac{a^2}{r^2}}{1 - \frac{a}{r}[e^{i(\theta-\phi)} + e^{-i(\theta-\phi)}] + \frac{a^2}{r^2}} \frac{h(\phi)}{2\pi}\, d\phi \\
&= \int_0^{2\pi} \frac{r^2 - a^2}{r^2 - 2ar\frac{e^{i(\theta-\phi)} + e^{-i(\theta-\phi)}}{2} + a^2} \frac{h(\phi)}{2\pi}\, d\phi
\end{aligned}$$

Therefore, the Poisson formula for the exterior of a circle $(r > a)$ is

$$u(r,\theta) = (r^2 - a^2) \int_0^{2\pi} \frac{h(\phi)}{r^2 - 2ar\cos(\theta - \phi) + a^2} \frac{d\phi}{2\pi}.$$